Last time: • smooth atlases $\mathcal{A}_1, \mathcal{A}_2$ on M are compatible ($\mathcal{A}_1 \sim_{C^\infty} \mathcal{A}_2$) if
$\mathcal{A}_1 \cup \mathcal{A}_2$ is a smooth atlas $\iff$ transition functions between $\mathcal{A}_1$ & $\mathcal{A}_2$ are smooth.
• each equivalence class (under $\sim_{C^\infty}$) of smooth atlas has a canonical representative which is maximal, $\mathcal{A}_{max} \in [\mathcal{A}]$. $\mathcal{A} \sim_{C^\infty} \mathcal{A}' \iff \mathcal{A}_{max} = (\mathcal{A}')_{max}$..
Call $\mathcal{A}_{max}$ / $[\mathcal{A}]$ the differentiable structure on M.

Ex: V n-dim'l vector space. Using a linear isomorphism $T: V \xrightarrow{\cong} \mathbb{R}^n$, we defined an atlas
$\mathcal{A}_T = \{U_T = V, \phi_T = T : V \xrightarrow{\cong} \mathbb{R}^n\}$. If T' is a different linear iso, observe that
$\mathcal{A}_T \sim_{C^\infty} \mathcal{A}_{T'}$ because

$$\begin{array}{ccc} \phi_{T'}(U_{T'}) & \xrightarrow{\phi_T \circ \phi_{T'}^{-1}} & \phi_T(U_T) \\ \parallel & & \parallel \\ \mathbb{R}^n & \xrightarrow{T' \circ T} & \mathbb{R}^n \end{array}$$

is C^∞; $T' \circ T$ is a linear isomorphism,
$\Rightarrow$ V has a canonical differentiable structure, call it the standard/linear differentiable structure.

From last time:
Functions: $(M, \mathcal{A} = \{U_\alpha, \phi_\alpha : U_\alpha \to \mathbb{R}^n\})$ smooth manifold of dimension n.
$f: M \to \mathbb{R}$ is smooth at $p \in M$ if $\exists$ $U_\alpha \ni p$ in $\mathcal{A}$ s.t.
$f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \longrightarrow \mathbb{R}$ is smooth at $\phi_\alpha(p)$.
$\cap$ open $\mathbb{R}^n$

(*)
$\iff$ (last time) for any $U_\beta \ni p$ in $\mathcal{A}$, $f \circ \phi_\beta^{-1} : \phi_\beta(U_\beta) \to \mathbb{R}$ is smooth at $\phi_\beta(p)$.

$f: M \to \mathbb{R}$ is smooth if it's smooth at every $p \in M$ $\iff$ for every $U_\alpha \in \mathcal{A}$,
$f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \longrightarrow \mathbb{R}$ is smooth.

Write $C^\infty_{(\mathcal{A})}(M)$ for the ~~set~~ algebra of smooth functions $M \to \mathbb{R}$.
(check: • if $f: M \to \mathbb{R}$ smooth then $cf : M \to \mathbb{R}$ smooth
• $f, g: M \to \mathbb{R}$ smooth then $f+g$ is also smooth, as is $f \cdot g$.
-- axioms of an $\mathbb{R}$-algebra.)

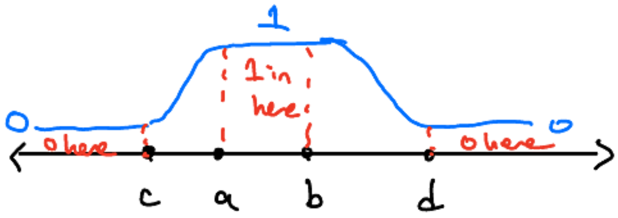
A priori; $C^\infty_{(\mathcal{A})}(M)$ depends on $\mathcal{A}$. However, we'll prove:

Lemma: $\mathcal{A}_1 \sim_{C^\infty} \mathcal{A}_2$ (i.e., $\mathcal{A}_1$ and $\mathcal{A}_2$ represent the same diff. structure) if and only if $\quad C^\infty_{(\mathcal{A}_1)}(M) = C^\infty_{(\mathcal{A}_2)}(M)$ $\quad$ (as subsets of $C^0(M)$)

The proof of one direction of this uses the following standard fact about $C^\infty(\mathbb{R}) / C^\infty(\mathbb{R}^n)$:

(exercise)

Lemma: ['C^∞ bump functions']: There exist C^∞ functions $h: \mathbb{R} \to [0,1]$ which equal 1 on $[a,b]$ and 0 on $\mathbb{R} \setminus [c,d]$ for any $c < a < b < d$.



(exercise)

$\Longrightarrow$ $\exists$ C^∞ functions $\mathbb{R}^n \to \mathbb{R}$ which equal 1 on $B_{r_1}(x_0)$ and 0 outside $B_{r_2}(x_0)$ for any $r_1 < r_2$.

Proof of:

Lemma: $\mathcal{A}_1 \sim_{C^\infty} \mathcal{A}_2$ (i.e., $\mathcal{A}_1$ and $\mathcal{A}_2$ represent the same diff. structure)

if and only if $\quad C^\infty_{(\mathcal{A}_1)}(M) = C^\infty_{(\mathcal{A}_2)}(M)$ $\quad$ (as subsets of $C^0(M)$)

Pf: $\Longrightarrow$ Say $\mathcal{A}_1 \sim_{C^\infty} \mathcal{A}_2$, meaning $\mathcal{A}_1 \cup \mathcal{A}_2$ is a smooth atlas.

By symmetry, we'll just show $C^\infty_{(\mathcal{A}_1)}(M) \subseteq C^\infty_{(\mathcal{A}_2)}(M)$.

Let $f \in C^\infty_{\mathcal{A}_1}(M)$. Meaning, $f: M \to \mathbb{R}$ is smooth at every pt w.r.t. $\mathcal{A}_1$ e.g., at every p, $\exists$ $U_\alpha \in \mathcal{A}_1 \subseteq \mathcal{A}_1 \cup \mathcal{A}_2$ containing p s.t. $f \circ \phi_\alpha^{-1}$ is smooth at $\phi_\alpha(p)$.

$\Longrightarrow$ (by (*) applied to $\mathcal{A}_1 \cup \mathcal{A}_2$) for any $U_\beta \in \mathcal{A}_1 \cup \mathcal{A}_2$ & in particular for any $U_\beta \in \mathcal{A}_2$ containing p, $f \circ \phi_\beta^{-1}$ is smooth at $\phi_\beta(p)$

$\Rightarrow f \in C^\infty_{\mathcal{A}_2}(M)$.

$\Longleftarrow$ Let's say $C^\infty_{\mathcal{A}_1}(M) = C^\infty_{\mathcal{A}_2}(M)$ for a pair of smooth atlases $\mathcal{A}_1, \mathcal{A}_2$.

We need to show $\mathcal{A}_1 \sim_{C^\infty} \mathcal{A}_2$ i.e., that $\mathcal{A}_1 \cup \mathcal{A}_2$ is a smooth atlas, or equivalently that for any

$$(U_\alpha, \phi_\alpha) \in \mathcal{A}_1, \quad (V_\beta, \psi_\beta) \in \mathcal{A}_2, \text{ that}$$

$$\psi_\beta \circ \phi_\alpha^{-1} : \underbrace{\phi_\alpha(U_\alpha \cap V_\beta)}_{\substack{\cap \text{ open} \\ \mathbb{R}^n}} \longrightarrow \underbrace{\psi_\beta(U_\alpha \cap V_\beta)}_{\substack{\cap \text{ open} \\ \mathbb{R}^n}} \xrightarrow{x_j} \mathbb{R} \text{ is smooth.}$$

Post composing with $x_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ (jth coord fcn), equivalently we must show that $x_j \circ \psi_\beta \circ \phi_\alpha^{-1} = (\psi_\beta \circ \phi_\alpha^{-1})_j$ is smooth for each j, at every $\phi_\alpha(p)$, $p \in U_\alpha \cap V_\beta$.

Fix $p \in U_\alpha \cap V_\beta$.

Above, have

$$x_j : \underset{\substack{\psi \\ \psi_\beta(p)}}{\psi_\beta(V_\beta)} \longrightarrow \mathbb{R}$$

By bump function Lemma, we can write down a function

$$\beta \cdot x_j : \psi_\beta(V_\beta) \longrightarrow \mathbb{R} \quad \text{which} \quad \begin{Bmatrix} \text{equals } x_j \text{ in } B_{\varepsilon/2}(\psi_\beta(p)) \\ \text{equals } 0 \text{ outside } B_\varepsilon(\psi_\beta(p)) \end{Bmatrix}$$

ε small enough so this ball lies in $\psi_\beta(V_\beta)$.

Now, note that

open subset of M.

$$(\beta \cdot x_j) \circ \psi_\beta : V_\beta \longrightarrow \mathbb{R}$$

extends to a function $\tilde{x}_j$ defined on all of M by setting this function to be 0 outside V_β; &

Claim: $\tilde{x}_j$ is smooth with respect to $\mathcal{A}_2$ (exercise: follows from fact that $\beta x_j := \tilde{x}_j \circ \psi_\beta^{-1}$ is smooth on $\psi_\beta(V_\beta)$).

$$\underbrace{\psi_\beta(V_\beta) \xrightarrow{\psi_\beta^{-1}} M \xrightarrow{\tilde{x}_j}}_{\beta x_j} \mathbb{R}$$

$$\Rightarrow \tilde{x}_j \in C^\infty_{\mathcal{A}_2}(M) \xRightarrow{\boxed{\text{hypothesis}}} \tilde{x}_j \in C^\infty_{\mathcal{A}_1}(M)$$

$$\underset{(\text{def'n})}{\Longrightarrow} \tilde{x}_j \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \longrightarrow \mathbb{R} \quad \text{is smooth.}$$

$$\|$$

$$\Rightarrow (\beta x_j) \circ \psi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap V_\beta) \xrightarrow{\psi_\beta \circ \phi_\alpha^{-1}} \psi_\beta(U_\alpha \cap V_\beta) \xrightarrow{\beta x_j} \mathbb{R}.$$

is smooth at $\phi_\alpha(p)$, but note in image of small nhood of $\phi_\alpha(p)$ under $\psi_\beta \circ \phi_\alpha^{-1}$ (i.e. in a small nhood of $\psi_\beta(p)$), $\beta \equiv 1$.

$\Rightarrow x_j \circ \psi_\beta \circ \phi_\alpha^{-1}$ is smooth at $\phi_\alpha(p)$.

But, α, β, p arbitrary, as was j $\Rightarrow \mathcal{A}_1 \cup \mathcal{A}_2$ is a smooth atlas. $\blacksquare$.

Operations on functions:

Topological spaces: Let's say $\phi: X \to Y$ is a continuous (C^0) map of top. spaces, then $\exists$ naturally defined pullback map

$$\phi^*: C^0(Y) \longrightarrow C^0(X)$$

$C^0(X) := \{\text{continuous maps } X \to \mathbb{R}\}$

$$f \longmapsto f \circ \phi$$

(put $(-)^{op}$ next to domain of functor)

$\Rightarrow$ "The functor $\mathrm{Top}^{op} \xrightarrow{C^0(-)} \mathrm{Set}/\mathrm{Alg}_{\mathbb{R}}$ is a contravariant functor, i.e.,

$X \longmapsto C^0(X)$

morphisms X to Y induce morphisms $C^0(Y) \to C^0(X)$."

Now say $(M, \mathcal{A})$ smooth manifold, and $\psi: M \to M$ a homeomorphism (of top. spaces). Then $\psi^*: C^0(M) \xrightarrow{\cong} C^0(M)$

$$\begin{matrix} C^0(M) \\ \cup| \\ C^\infty_{[\mathcal{A}]}(M) \end{matrix}$$

It's not true in general that $\psi^*(C^\infty_{(\mathcal{A})}(M)) = C^\infty_{(\mathcal{A})}(M)$; in fact the two might be distinct.

Def: Two C^∞ structures $C^\infty_{\mathcal{A}_1}(M)$ and $C^\infty_{\mathcal{A}_2}(M)$ are equivalent if $\exists$ a homeomorphism ψ of M taking $C^\infty_{\mathcal{A}_1}(M) \xrightarrow{\sim} C^\infty_{\mathcal{A}_2}(M)$.

Amazing fact: [Milnor, ...]: S^7 has several inequivalent smooth structures!

(in contrast, S^1, S^2, S^3 have only one, S^4 unknown [Smooth Poincaré conj.])