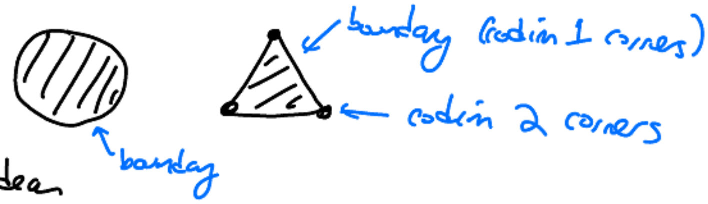
Last time: manifolds M & functions (smooth) $f: M \longrightarrow \mathbb{R}$

Before continuing this discussion: enlarge the class of manifolds (sometimes) to allow <u>boundary</u> (<u>corners</u>). e.g.,



features: • at interior points, locally Euclidean

• at boundary points, locally modeled on $\mathbb{H}^n = \{(x_1, \dots, x_n) \mid x_1 \le 0\}$

(corners: local model is an octant in $\mathbb{R}^n$).

<u>Def</u>: A (smooth) <u>m'fold-with-boundary</u> <u>of dimension</u> m, M is a pair $M := (M, \mathcal{A})$ where M is Hausdorff, second countable top. space, and

$$\mathcal{A} = \{(U_\alpha, \phi_\alpha : U_\alpha \xrightarrow{\text{homeo.}} \phi_\alpha(U_\alpha) \underset{\text{open}}{\subseteq} \mathbb{H}^m\}$$

(existence of $\mathcal{A}$ w/o transition fcn. smoothness: "top. manifold w/ boundary")

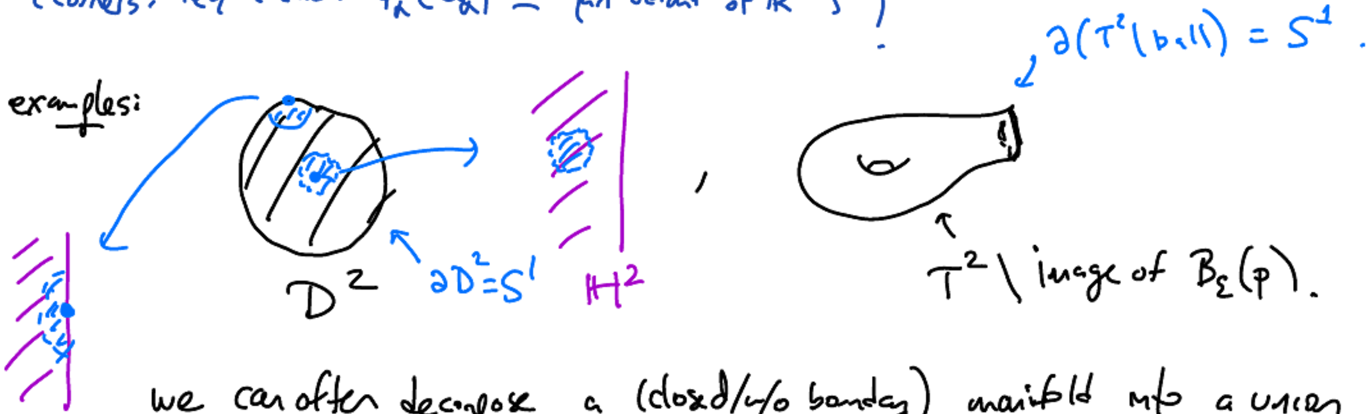
s.t. the transition functions

$$\phi_\alpha \circ \phi_\beta^{-1} : \underset{\substack{\cap|\text{ open} \\ \mathbb{H}^m}}{\phi_\beta(U_\alpha \cap U_\beta)} \longrightarrow \underset{\substack{\cap| \\ \mathbb{H}^m}}{\phi_\alpha(U_\alpha \cap U_\beta)}$$

are smooth.

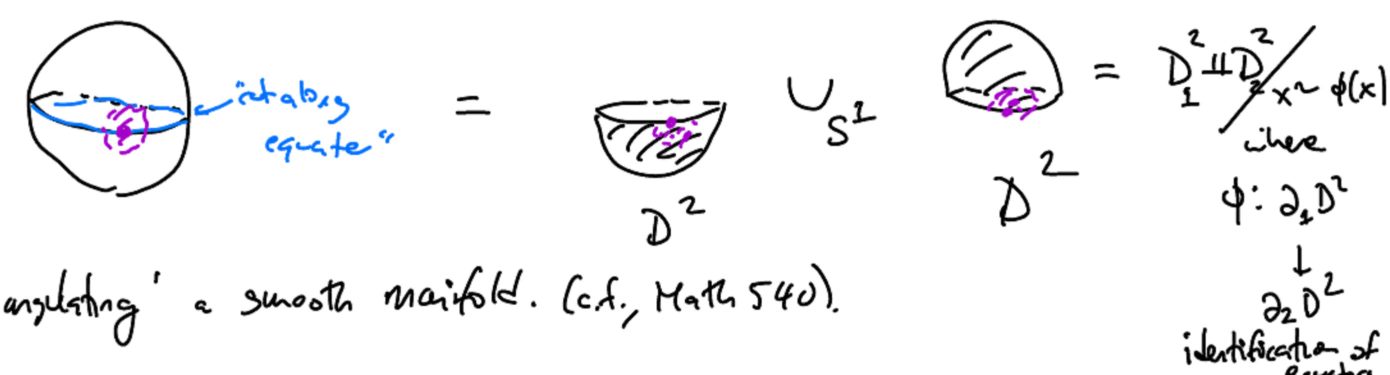
(corners: require each $\phi_\alpha(U_\alpha) \overset{\text{open}}{\subseteq} \{\text{an octant of } \mathbb{R}^m\}$)

examples:



we can often decompose a (closed/w/o boundary) manifold into a union of manifolds w/ boundary (or corners)

e.g.,



<u>OR</u>: 'triangulating' a smooth manifold. (c.f., Math 540).

Def: The boundary of M, denoted ∂M, is the set of points of M which lie in the boundary of $\mathbb{H}^n$ under some chart map ϕ_α.

Lemma: ("invariance of boundary"): if some chart around p takes p to $\partial \mathbb{H}^m$, then every chart around p does. ($\Rightarrow$ ∂M is well-defined).

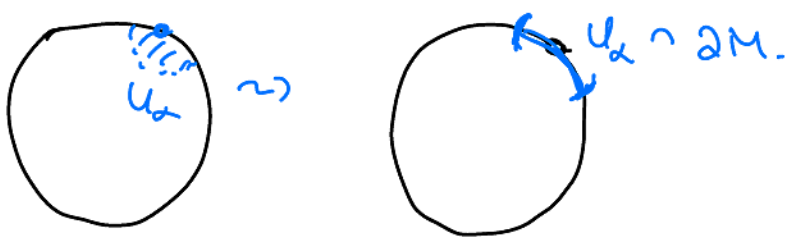
boundary of M, ∂M $\Longleftrightarrow$ set of non-interior points of M.

Lemma: ∂M is ~~an~~ a smooth $(m-1)$-manifold.

Pf sketch: Take maximal atlas associated to M, restrict to those (U_α, ϕ_α) covering boundary points & restrict to the boundary via $(U_\alpha \cap \partial M, \phi_\alpha|_{U_\alpha \cap \partial M})$

has image in $\partial \mathbb{H}^m \cong \mathbb{R}^{m-1}$

Picture:



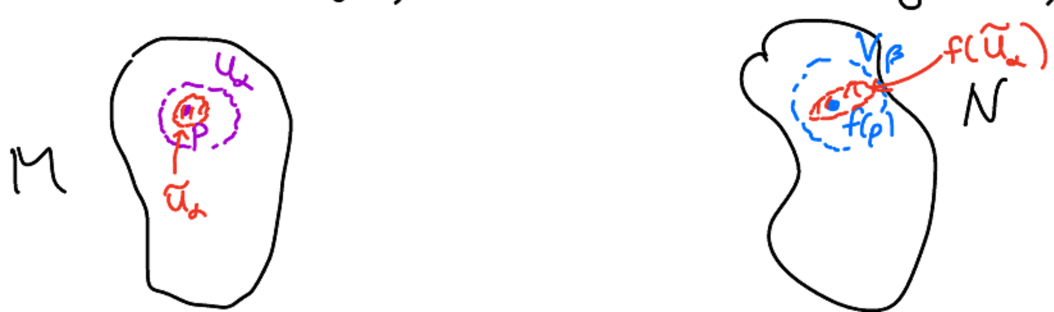
... (exercise). $\square$.

Many notions covered in class will be stated for smooth manifolds but extend to the case w/ boundary/corners, e.g., $\exists$ $C^\infty(M)$ m'fold-with-∂.

Note: the complement $M \setminus \partial M$ is a manifold (w/o boundary), may be non-compact if ∂M was non-empty.

Smooth maps $f: M \to N$ (for simplicity, assume M, N have no boundary, but notion extends)

Def: A continuous map $f: M^m \to N^n$ is smooth at $p \in M$ if there exists $(U_\alpha, \phi_\alpha) \in \mathcal{A}_M$ containing p, $(V_\beta, \psi_\beta) \in \mathcal{A}_N$ containing $f(p)$, such that:



for some open subset $\tilde{U}_\alpha \subseteq U_\alpha$ containing p & small enough so that $f(\tilde{U}_\alpha) \subseteq V_\beta$,

(exists b/c f is continuous!)

the composition

$$\phi_\alpha(p) \longmapsto p.$$

$$\begin{array}{ccccccc} \phi_\alpha(\tilde{U}_\alpha) & \xrightarrow[\cong]{\phi_\alpha^{-1}} & \tilde{U}_\alpha & \xrightarrow{f} & V_\beta & \xrightarrow[\cong]{\psi_\beta} & \psi_\beta(V_\beta) \\ \cap\!| \text{ open} & & & & & & \cap\!| \text{ open} \\ \mathbb{R}^m & & & & & & \mathbb{R}^n \end{array}$$

is smooth at the point $\phi_\alpha(p)$. (note: this notion doesn't depend on the shrinking $p \in \tilde{U}_\alpha \underset{\text{open}}{\subseteq} U_\alpha$ w/ $f(\tilde{U}_\alpha) \subseteq V_\beta$)

As before, this condition is true for one set of charts around p & $f(p)$

$\iff$ this condition holds for any choice of charts around p & $f(p)$

(exercise).

$f: M \to N$ is <u>smooth</u> if it's smooth at every p.

(e.g., $x^2 : \mathbb{R}^2_{(x,y)} \to \mathbb{R}$ is a smooth map)

<u>Lemma</u>: A continuous map $f: M \to N$ is smooth if and only if

$f^*(C^\infty(N)) \subset C^\infty(M)$.

Pf of $\Rightarrow$: If $g: N \to \mathbb{R}$ is smooth, check that $g \circ f = f^* g : M \to \mathbb{R}$ is smooth too. (More general fact is that compositions of smooth maps are smooth)

$\Leftarrow$ <u>exercise</u>. ∎

<u>Def</u>: A smooth map $f: M \to N$ is a <u>diffeomorphism</u> if $\exists$ a smooth two-sided inverse

$f^{-1}: N \to M$ (e.g., $f^{-1} \circ f = id_M$, $f \circ f^{-1} = id_N$)

(note: $id_M : M \to M$ is smooth)

<u>Some abuses of notation that are common</u>:

- $M := (M, \mathcal{A})$ / rather $(M, \mathcal{A}_{max}$ or $[\mathcal{A}])$.
- given a chart (U_α, ϕ_α), the map $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \underset{\text{open}}{\subseteq} \mathbb{R}^n$ gives "<u>local coordinates</u>", e.g., the functions $x_1, \ldots, x_n : \mathbb{R}^n \to \mathbb{R}$. ~~the~~ give functions

$x_1 \circ \phi_\alpha, \dots, x_n \circ \phi_\alpha$ on U_α, called <u>local coordinates</u>.

(often abbreviated $x_1, \dots, x_n : U_\alpha \to \mathbb{R}$)

By ~~post~~ composing a given ϕ_α with a diffeomorphism (giving another element $(U_\alpha, \overbrace{f \circ \phi_\alpha}^{\tilde{\phi}_\alpha})$ of $\mathcal{A}_{max}$),

can find a chart sending a given $p \in U_\alpha$ to origin in $\mathbb{R}^n$.

The resulting local coordinates ~~$x_1 \circ \tilde{\phi}_\alpha, \dots, x_n \circ \tilde{\phi}_\alpha$~~ $x_1, \dots, x_n$ are called "local coordinates centered at p; all 0 at p."