Last time: smooth maps and diffeomorphisms = smooth maps w/ smooth inverses.

Towards the derivative of a smooth map.

$f: M^m \longrightarrow N^n$ smooth. Pick a point $p \in M$, and charts

(U_α, ϕ_α) of M and (V_β, ψ_β) of N with $f(U_\alpha) \subseteq V_\beta$

$U_\alpha \ni p$, $V_\beta \ni f(p)$

$\rightsquigarrow$ the map

$$\psi_\beta \circ f \circ \phi_\alpha^{-1} : \underbrace{\phi_\alpha(U_\alpha)}_{\substack{\cap \text{ open} \\ \mathbb{R}^m}} \longrightarrow \underbrace{\psi_\beta(V_\beta)}_{\substack{\cap \text{ open.} \\ \mathbb{R}^n}}$$

is a C^∞ map between subsets of Euclidean space, so we can take its derivative:

$$d(\psi_\beta \circ f \circ \phi_\alpha^{-1})(x) : \mathbb{R}^m \longrightarrow \mathbb{R}^n, \text{ in particular at } x = \phi_\alpha(p).$$

The map $d(\psi_\beta \circ f \circ \phi_\alpha^{-1})(\phi_\alpha(p))$ depends on the charts but has certain properties that only depend on p, namely its rank.

want to call 'the derivative at p of f', but it depends on choices!

Recall: V (fin dim'l) vector space $(/\mathbb{R})$, $\dim V = \#$ elts. in a basis of V,

and for a linear map $T: V \to W$, $\operatorname{rank}(T) := \dim(\operatorname{im}(T))$

Note: $\operatorname{rank}(T) \leq \min(\dim V, \dim W)$.

Def: The rank of a smooth (not nec. linear) map $f: \mathbb{R}^m \to \mathbb{R}^n$ (or $f: \underset{\mathbb{R}^m}{U} \to \underset{\mathbb{R}^n}{V}$) at $x \in \mathbb{R}^m$ is the rank of the linear map $df(x): \mathbb{R}^m \to \mathbb{R}^n$. f has constant rank (over U) if $df(x)$ has constant rank as one varies x. (in U)

Lemma: $f: M^m \to N^n$ smooth as above, pick p as above too. Then the quantity $\operatorname{rank}(d(\psi_\beta \circ f \circ \phi_\alpha^{-1})(\phi_\alpha(p)))$ as defined above for (U_α, ϕ_α) in M and (V_β, ψ_β) (with $f(U_\alpha) \subseteq V_\beta$), $U_\alpha \ni p$, $V_\beta \ni f(p)$, is independent of (U_α, ϕ_α), (V_β, ψ_β). Call it the rank of f at p, denoted $\operatorname{rk}_p(f)$.

Pf sketch: Let $(\tilde{U}_\alpha, \tilde{\Phi}_\alpha)$, $(\tilde{V}_\beta, \tilde{\Psi}_\beta)$ another choice of charts. WLOG, $\tilde{U}_\alpha \ni p$, $\tilde{V}_\beta \ni f(p)$

shrink these nhoods and (U_α,ϕ_α), (V_β,ψ_β) so that $\tilde{U}_\alpha \subseteq U_\alpha$, $V_\beta \subseteq \tilde{V}_\beta$, so have

maps $\quad \tilde{U}_\alpha \subseteq U_\alpha \xrightarrow{f} V_\beta \subseteq \tilde{V}_\beta$.

picture w/ chart maps:

$$\begin{tikzcd}
\tilde{\phi}_\alpha(\tilde{U}_\alpha) \arrow[r, "\tilde{\phi}_\alpha^{-1}"] \arrow[rrr, bend left, "\tilde{\psi}_\beta\circ f\circ\tilde{\phi}_\alpha^{-1}"] & \tilde{U}_\alpha \arrow[r, "f"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{V}_\beta \arrow[r] & \tilde{\psi}_\beta(\tilde{V}_\beta) \\
 & U_\alpha \arrow[dl, "\phi_\alpha"'] \arrow[d, equal] & V_\beta \arrow[u, hook] \arrow[d, equal] & \\
\phi_\alpha(U_\alpha) \arrow[r, "\phi_\alpha^{-1}"'] & U_\alpha \arrow[r, "f"] & V_\beta \arrow[r, "\psi_\beta"'] & \psi_\beta(V_\beta) \arrow[ul, "\psi_\beta^{-1}"']
\end{tikzcd}$$

$$\Rightarrow (\tilde{\psi}_\beta\circ f\circ\tilde{\phi}_\alpha^{-1}) = \underbrace{\tilde{\psi}_\beta\circ\psi_\beta^{-1}}_{\substack{\text{transition function for}\\ (V_\beta,\psi_\beta),\ (\tilde{V}_\beta,\tilde{\psi}_\beta).}} \circ(\psi_\beta\circ f\circ\phi_\alpha^{-1})\circ\underbrace{\phi_\alpha\circ\tilde{\phi}_\alpha^{-1}}_{\substack{\text{transition function for}\\ (U_\alpha,\phi_\alpha)\ \ (\tilde{U}_\alpha,\tilde{\phi}_\alpha)}}.$$

The chain rule implies:

$$d(\tilde{\psi}_\beta\circ f\circ\tilde{\phi}_\alpha^{-1})_{\tilde{\phi}_\alpha(p)} = d(\tilde{\psi}_\beta\circ\psi_\beta^{-1})_{\psi_\beta(f(p))}\circ d(\psi_\beta\circ f\circ\phi_\alpha^{-1})_{\phi_\alpha(p)}\circ d(\phi_\alpha\circ\tilde{\phi}_\alpha^{-1})_{\tilde{\phi}_\alpha(p)}$$

transition functions are diffeos, so $d(\text{transition fcn.})$ is a linear iso.

Now note that if $T: V\longrightarrow W$ any linear map,

$S_1: V\xrightarrow{\cong} V$, $\quad S_2: W\xrightarrow{\cong} W$, then $\operatorname{rank}(T) = \operatorname{rank}(S_2\circ T\circ S_1)$.

So $\operatorname{rank}\left(d(\tilde{\psi}_\beta\circ f\circ\tilde{\phi}_\alpha^{-1})_{\tilde{\phi}_\alpha(p)}\right) = \operatorname{rank}\left(d(\psi_\beta\circ f\circ\phi_\alpha^{-1})_{\phi_\alpha(p)}\right)$. $\boxtimes$

This gives evidence for the idea that there should be a canonical notion of <u>derivative of f at p</u>, for $f: M \to N$, $p \in M$, as a linear map from some m-dim'l vector space associated to (M, p) to some n-dim'l vector space associated to $(N, f(p))$, so that a chart around (M,p) identifies the first vector space w/ $\mathbb{R}^m$, & a chart around $(N, f(p))$ identifies the second vector space w/ $\mathbb{R}^n$.

We can implement this idea; first we must define the notion of a <u>tangent space to a manifold at a point</u>, this will give the vector space needed above.

Tangent vectors:

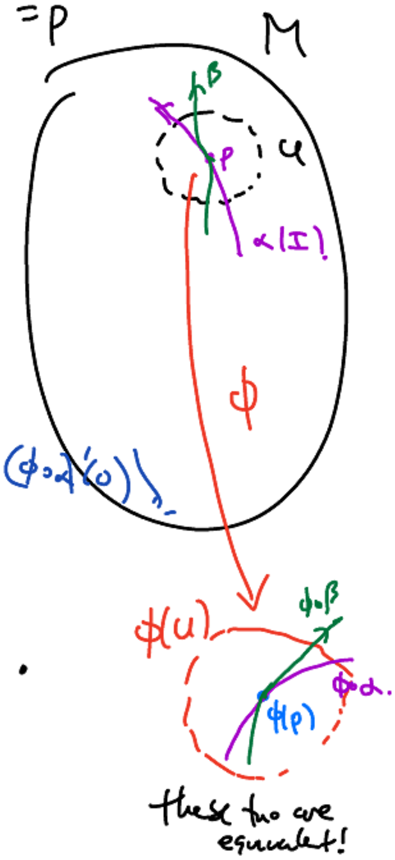
I. Tangent vectors as equivalence classes of smooth curves

<u>Def</u>: A <u>parametrized curve</u> in a C^∞ manifold M is a <u>smooth</u> map $I \to M$, where $I \subseteq \mathbb{R}$ open interval

If M^m smooth manifold, $p \in M$, define $C_p := \{$parametrized curves $\alpha: I \to M$ with $I \ni 0$ and $\alpha(0) = p\}$.

Choose a chart (U, ϕ) of $\mathcal{A}_{max}$ on M with $p \in U$.

Since U is open, any parametrized curve $\alpha: I \to M$ w/ $\alpha(0) = p$ can be restricted to $\underset{\substack{\psi \\ 0}}{I'} \subseteq I$, w/ $\alpha(I') \subseteq U$, after which one can use ϕ to associate to $\alpha|_{I'}$ the curve



$$\begin{aligned} \phi \circ \alpha : I' &\longrightarrow \phi(U) \underset{\text{open}}{\subseteq} \mathbb{R}^m. \\ 0 &\longmapsto \phi(p). \end{aligned}$$

(note $\phi \circ \alpha$ is smooth at 0 b/c α differentiable $\Rightarrow$ have $(\phi \circ \alpha)'(0)$).

Using the chart (U, ϕ), define $\sim$ on C_p by:

$$\alpha \sim \beta \text{ exactly when } (\phi \circ \alpha)'(0) = (\phi \circ \beta)'(0).$$

<u>Lemma</u>: The equivalence relation $\sim$ on C_p is independent of

choice of chart (U, ϕ) containing p.

Pf: (exercise).

Def: The tangent space T_pM of M at p is the set of equivalence classes $C_p/\sim$. Call an element of T_pM a tangent vector to M at p.

(not obvious from def'n: this is a vector space.. Next time: state results which show T_pM has a canonical vec. space structure, & give other definitions of T_pM where this structure is immediately apparent).