From last time:

If M^m smooth manifold, $p \in M$, define $C_p := \{$parametrized curves $\alpha : I \to M$ with $I \ni 0$ and $\alpha(0) = p\}$.

Def: The tangent space T_pM of M at p is the set of equivalence classes $C_p/\sim$. Call an element of T_pM a tangent vector to M at p.

where $\alpha \sim \tilde{\alpha}$ if $(\phi \circ \alpha)'(0) = (\phi \circ \tilde{\alpha})'(0)$ for one equivalently any chart (U, ϕ) containing p.

Lemma: (1) If (U, ϕ) chart containing p, then the map

(exercises)

$$\Phi : T_pM \longrightarrow \mathbb{R}^m \quad \text{defined by}$$

$$[\alpha] \longmapsto (\phi \circ \alpha)'(0) \quad \text{is a bijection.}$$

(2) For any other chart $(\tilde{U}, \tilde{\phi})$ around p, the map

$$\tilde{\Phi} \circ (\Phi^{-1}) : \mathbb{R}^m \xrightarrow{\Phi^{-1}} T_pM \xrightarrow{\tilde{\Phi}} \mathbb{R}^m$$

coincides with $d(\underbrace{\tilde{\phi} \circ \phi^{-1}}_{\text{transition fn (diffeo.)}})(\phi(p))$, which is a linear isomorphism,

in particular it's linear.

Cor: The tangent space T_pM has a unique structure as a vector space over $\mathbb{R}$ s.t. for every (U, ϕ) around p, the map $\Phi : T_pM \to \mathbb{R}^m$ is a linear iso.

Notation: If $v \in T_pM$ is the vector rep. by a curve $\alpha : I \to M$ w/ $\alpha(0) = p$, we'll say that "v is the vector tangent to α at $t = 0$ (or at the point p)" and write "$v = \alpha'(0)$."

II. Tangent spaces as spaces of derivations.

A. Some 'sheaf-theoretic' notions:

- $\underset{\text{open set}}{U \subseteq M} \underset{\text{can define}}{\rightsquigarrow} C^\infty(U) := \{ \text{smooth functions } f: U \to \mathbb{R} \}$

  (recall $U \subseteq_{\text{open}} M$ is naturally an m-m'fold too).

  $C^\infty(U)$ is an $\mathbb{R}$-<u>algebra</u>: operations cf, $f \cdot g$, $f+g$ for any $c \in \mathbb{R}$, $f, g \in C^\infty(U)$.

- Given $U_1 \subseteq U_2$ open sets in M, there's a natural restriction map

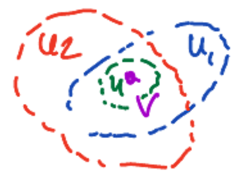
$$\rho^{U_2}_{U_1} : C^\infty(U_2) \longrightarrow C^\infty(U_1)$$
$$f \longmapsto f|_{U_1}.$$

(note: for $U_1 \subseteq U_2 \subseteq U_3$, $\rho^{U_2}_{U_1} \rho^{U_3}_{U_2} = \rho^{U_3}_{U_1}$).

"$C^\infty(-)$ is a <u>presheaf</u> on M, meaning a functor $\{\text{open sets in } M, \subseteq\}^{op} \longrightarrow \text{algebras}/\mathbb{R}$"

(see more of this on HW: in fact $C^\infty(-)$ is a <u>sheaf</u>, which is a presheaf satisfying certain nice "local-to-global" properties)

- $V \subseteq M^m$ any subset, not necessarily open. Then, we can define



$V = \{p\}$

$$C^\infty(V) := \left\{ (f, U) \;\middle|\; \begin{array}{l} U \subseteq_{\text{open}} M, \; V \subset U, \\ f \in C^\infty(U) \end{array} \right\} \Big/ \sim$$

where $(f_1, U_1) \sim (f_2, U_2)$ if there exists

$$V \subset \underset{\text{open}}{U} \subset U_1 \cap U_2 \text{ on which } f_1|_U = f_2|_U.$$

<u>exercise</u>: $C^\infty(V)$ is still an $\mathbb{R}$-algebra.

Note: Algebraically, can write $C^\infty(V)$ as the "direct limit"

$$C^\infty(V) := \varinjlim_{\substack{U \supseteq V \\ \text{open}}} C^\infty(U).$$

A "smooth function on V", i.e., an element of $C^\infty(V)$, is more aptly sometimes called a "germ of a smooth function on V."

Main example: $V = \{p\} \subseteq M \rightsquigarrow C^\infty(\{p\})$ (or just $C^\infty(p)$) is the set of germs of smooth functions on p.

(Note: $C^\infty(p)$ is the "stalk at p" of the (pre-)sheaf $C^\infty(-)$ on M).

B. The definition: M^m mfold.

Def: A derivation (at p) is an $\mathbb{R}$-linear map

$$X : C^\infty(p) \longrightarrow \mathbb{R}$$

obs: an element $f \in C^\infty(p)$ has well-defined value $f(p) \in \mathbb{R}$

satisfying the Leibniz rule: $X(fg) = X(f)\underbrace{g(p)}_{\in \mathbb{R}} + \underbrace{f(p)}_{\in \mathbb{R}}X(g).$

Def 2 of tangent space:

The tangent space $T_pM := \{\text{derivations } C^\infty(p) \longrightarrow \mathbb{R}\}$

Elements of T_pM, i.e., derivations, are called tangent vectors.

Note: w.r.t. vector space structure on $\mathrm{Hom}_{\mathrm{Vect}_\mathbb{R}}(C^\infty(p), \mathbb{R})$, derivations are a subspace, in particular T_pM is naturally an $\mathbb{R}$-vector space, defined this way.

Example of a derivation: $M = \mathbb{R}^m$, $p \in \mathbb{R}^m$, & consider the directional derivative $D_{\vec{v}}$ at p along a tangent vector $\vec{v} \in \mathbb{R}^m$, defined by $D_{\vec{v}}(f)(p) = (f \circ \alpha)'(0)$ for any curve $\alpha : I \to \mathbb{R}^m$ with $\alpha(0) = p$, $\alpha'(0) = v$.

check: (1) $D_{\vec{v}}(-)(p)$ gives a well-defined map $C^\infty(p) \longrightarrow \mathbb{R}$. ($\mathbb{R}$-linear).

(2) $D_{\vec{v}}(fg)(p) = (D_{\vec{v}}f)(p)\cdot g(p) + f(p)(D_{\vec{v}}g)(p)$.

Exercise: If $X : C^\infty(p) \to \mathbb{R}$ is a derivation, then $X(c) = 0$ for any (equiv. class of) constant function $c \in C^\infty(p)$

For the next example, note that since $C^\infty(p)$, $p \in M$, only depends on a sufficiently small nhood $\underset{p}{\underset{\Psi}{U}} \subseteq M$, it follows that any chart $(\underset{p}{\underset{\Psi}{U}}, \phi)$ induces an identification

$$C^\infty(p) \cong C^\infty(\phi(p))$$

$$\text{"} [f] \longmapsto [f \circ \phi^{-1}] \text{"}$$

Examples:

(1) For a set of local coords $x_1, \dots, x_m$ near p, get a "tangent vector" $X_i = \frac{\partial}{\partial x_i}$. Meaning, have a chart (U, ϕ) of p sending p to $\stackrel{\phi}{\mapsto} 0 \in \mathbb{R}^m$, then under identification $C^\infty(p) \cong C^\infty(0 \in \mathbb{R}^m)$, take $X_i(f) = \frac{\partial f}{\partial x_i}(0)$.

(i.e., as $X_i(f) := \frac{\partial(f \circ \phi^{-1})}{\partial x_i}(0)$).

(2) next time: map between Def'n 1 of T_pM & Def. 2 of T_pM.