Last time: M^m m'fold, $p \in M$ point:

Def 2 of T_pM: $T_pM := \mathrm{Der}(C^\infty(p), \mathbb{R})$

& **Example:** (1) Picking local coords. at p (i.e., a chart (U, ϕ) inducing functions $x_1, \dots, x_m := x_1 \circ \phi, \dots, x_m \circ \phi$),

$\frac{\partial}{\partial x_i}(-)\big|_{x=p}$.

get $X_i := \frac{\partial}{\partial x_i} \in T_pM$,

an element of

rep. of $C^\infty(\phi(p))$

$x_i : U \longrightarrow \underset{\substack{\cap \text{ open} \\ \mathbb{R}^m}}{\phi(U)} \xrightarrow{x_i} \mathbb{R}$.

meaning

$$\underset{\substack{\cap \\ C^\infty(p)}}{X_i([f])} := \frac{\partial (f \circ \phi^{-1})}{\partial x_i}(\phi(p)).$$

Note: in $\mathbb{R}^m$, given $a \in \mathbb{R}^m$, $\frac{\partial}{\partial x_i}(-)\big|_{x=a} : C^\infty(a) \longrightarrow \mathbb{R}$ is a derivation.

(2) Given $\alpha : (-\varepsilon, \varepsilon) \longrightarrow M$ param. curve with $\alpha(0) = p$, and $[f] \in C^\infty(p)$,

$\rightsquigarrow$ define a derivation X_α by

rep. of $[f]$

$$X_\alpha([f]) := (f \circ \alpha)'(0)$$

note $f \circ \alpha : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$.

Exercises:

Check:

- $X_\alpha([f])$ only depends on $[f]$, so gives $X_\alpha : C^\infty(p) \longrightarrow \mathbb{R}$.

- X_α is a derivation.

- If $\alpha \sim \bar{\alpha}$ in the sense of def'n 1 of tangent space, (meaning $(\phi \circ \alpha)'(0) = (\phi \circ \bar{\alpha})'(0)$ for any chart (U, ϕ) around p), then $X_\alpha = X_{\bar{\alpha}}$

$\Rightarrow$ get a well-defined map

$$T_pM^{(\text{def } 1)} := C_p/\sim \xrightarrow{(*)} T_pM^{(\text{def } 2)} := \mathrm{Der}(C^\infty(p), \mathbb{R})$$

$$[\alpha] \longmapsto X_\alpha$$

$\leftarrow$ only depends on $[\alpha]$ by above.

Lemma: The two definitions of T_pM are equivalent, & in fact $(*)$ is a linear isomorphism.

It remains to check: (a) $(*)$ is a linear map.

(b) isomorphism (if know both def'ns are vector spaces of same dimension, injectivity or surjectivity is sufficient).

Most of this is an exercise, but regarding (b): we've shown $\dim(T_pM^{\text{def } 1}) = m$ (by identification with $\mathbb{R}^m$).

Let's do the same directly for $T_pM^{\text{def } 2}$:

<u>Lemma</u>: If $\dim(M) = m$, then $\dim T_pM$ (as defined via def. 2) $= m$.

<u>Proof</u>: Using a chart (U, ϕ) <u>centered at p</u>, get an identification

$$C^\infty(p) \overset{\text{alg.}}{\cong} C^\infty(0 \in \phi(U) \subseteq \mathbb{R}^m), \text{ hence an iso.}$$

$$[f] \longmapsto [f \circ \phi^{-1}]$$

$$T_pM = \operatorname{Der}(C^\infty(p), \mathbb{R}) \underset{\text{linear}}{\cong} \operatorname{Der}(C^\infty(\overset{\mathbb{R}^m}{0}), \mathbb{R}) \quad (\cong T_0\mathbb{R}^m).$$

$\Rightarrow$ it suffices to show $\dim(T_0\mathbb{R}^m) = m$.

Let $x_1, \ldots, x_m \in C^\infty(\mathbb{R}^m)$ be usual coord. fcns, & $[x_1], \ldots, [x_m] \in C^\infty(0)$ assoc. germs,

& consider $X_i = \frac{\partial}{\partial x_i} \in T_0\mathbb{R}^m$, $i = 1, \ldots, m$. (shorthand for $\frac{\partial}{\partial x_i}(-)(0)$).

Note $X_i([x_j]) = \delta_{ij}$ ← Kronecker δ: $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases}$;

and note in particular that $\{X_i\}_{i=1}^m$ are linearly independent.

So $\dim T_0\mathbb{R}^m \geq m$.

Let's show $X_1, \ldots, X_m$ span $T_0\mathbb{R}^m$. Consider $X \in T_0\mathbb{R}^m$ an arbitrary derivation,

& say $X([x_i]) = b_i$ for some $b_i \in \mathbb{R}$.

Consider $Y := X - \sum_j b_j X_j \in T_0\mathbb{R}^m$. $Y : C^\infty(0) \to \mathbb{R}$ derivation with $Y(x_i) = 0$ for all x_i.

<u>Ingredient</u>: <u>Taylor's theorem</u>:

Let $f : \underset{\overset{\cap\,\text{open}}{\mathbb{R}^m}}{U} \longrightarrow \mathbb{R}$ smooth fcn, $0 \in U$.

Then we can write $f(x) = a + \sum a_i x_i + \sum_{i,j} a_{ij}(x) x_i x_j$

on some open rectangle $(-\varepsilon,\varepsilon)\times\cdots\times(-\varepsilon,\varepsilon)$ in U containing 0,

where a, $\{a_i\}_{i=1}^m$ are constants and $\{a_{ij}(x)\}_{i,j=1}^m$ are C^∞ functions.

e.g., $f(x) = 1 + x + x^3$, then $a = 1$, $a_1 = 1$, and $a_{1,1} = x$.

$= 1 + 1\cdot x + (x)\cdot x^2$.

Given $[f] \in C^\infty(0)$, using Taylor's theorem we can write a representative f of $[f]$

as (shrinking domain of f if needed while still containing 0)

$$a + \sum_i a_i x_i + \sum_{i,j} a_{ij}(x) x_i x_j \quad \text{as above.} \quad (\star)$$

by Leibniz Rule.

Note that for any derivation Z, $Z(a) = 0$, $Z(a_{ij}(x)\, x_i x_j) = 0$.

Applying to $Z = Y$, we learn that for any $[f]$, using Taylor approx. as $(\star)$,

$$Y([f]) = Y\left(\sum a_i x_i\right) = \sum a_i Y(x_i) = 0 \quad \text{b/c } Y(x_i) = 0.$$

$\Rightarrow Y \equiv 0. \Rightarrow X = \sum b_j X_j$. So $X_1, \dots, X_m$ span $T_0\mathbb{R}^m$.

$\Rightarrow \dim T_0\mathbb{R}^m = \dim T_pM = m.$ $\blacksquare$

<u>Def. 3 of tangent space</u> : M^m, $p \in M$ as before, again begin with $C^\infty(p)$.

<u>Define</u>: $\mathcal{F}_p \subseteq C^\infty(p)$ to be germs of fcns. which are 0 at p.

$$= \{(U,f) \mid f(p) = 0\}/\sim$$

$\mathcal{F}_p$ is an <u>ideal</u> in the algebra $C^\infty(p)$, meaning its a linear subspace & if $f \in \mathcal{F}_p$, $g \in C^\infty(p)$, then $gf \in \mathcal{F}_p$.

Let $\mathcal{F}_p^2 \subset \mathcal{F}_p$ be the ideal in $C^\infty(p)$ <u>generated</u> by products of two elements in $\mathcal{F}_p$.

$$= \left\{ \sum_{i,j} f_{ij} \phi_i \phi_j \;\middle|\; \phi_i \in \mathcal{F}_p,\ \phi_j \in \mathcal{F}_p,\ f_{ij} \in C^\infty(p) \right\}.$$

Def 3: $T_pM := \left(\mathcal{F}_p / (\mathcal{F}_p)^2 \right)^*$ linear dual

Lemma: There's a canonical iso. $(T_pM)^{\text{def. 2}} \cong (T_pM)^{\text{def. 3}}$.

Sketch: Let $X : C^\infty(p) \to \mathbb{R}$ be a derivation. Then can restrict $X|_{\mathcal{F}_p} : \mathcal{F}_p \to \mathbb{R}$,

then check $(X|_{\mathcal{F}_p})|_{\mathcal{F}_p^2} : \mathcal{F}_p^2 \longmapsto 0$. (exercise).

$\rightsquigarrow$ get an induced map $\overline{X|_{\mathcal{F}_p}} : \mathcal{F}_p / \mathcal{F}_p^2 \longrightarrow \mathbb{R}$, i.e., $\overline{X|_{\mathcal{F}_p}} \in \left(\mathcal{F}_p / (\mathcal{F}_p)^2 \right)^*$.

$\rightsquigarrow$ get our map

$$\begin{array}{ccc} (T_pM)^{\text{def. 2}} & \longrightarrow & (T_pM)^{\text{def. 3}} \\ \| & & \| \\ \mathrm{Der}(C^\infty(p), \mathbb{R}) & & \left(\mathcal{F}_p / (\mathcal{F}_p)^2 \right)^* \\ \ni & & \\ X & \longmapsto & \overline{X|_{\mathcal{F}_p}}. \end{array}$$

.... next time. $\boxtimes$