From last time:

Def. 3 of tangent space: M^m, $p \in M$ as before, again begin with $C^\infty(p)$.

Define: $\mathcal{F}_p \subseteq C^\infty(p)$ to be germs of fcns. which are 0 at p.

$$\mathcal{F}_p = \{(U,f) \mid f(p)=0\}/\sim$$

$\mathcal{F}_p$ is an ideal in the algebra $C^\infty(p)$, meaning its a linear subspace & if $f \in \mathcal{F}_p$, $g \in C^\infty(p)$, then $gf \in \mathcal{F}_p$.

Let $\mathcal{F}_p^2 \subset \mathcal{F}_p$ be the ideal in $C^\infty(p)$ generated by products of two elements in $\mathcal{F}_p$.

$$\mathcal{F}_p^2 = \Big\{ \sum_{\substack{i,j \\ \text{finite}}} f_{ij}\phi_i\phi_j \;\Big|\; \phi_i \in \mathcal{F}_p,\ \phi_j \in \mathcal{F}_p,\ f_{ij} \in C^\infty(p) \Big\}.$$

Def 3: $T_pM := \left(\mathcal{F}_p / (\mathcal{F}_p)^2 \right)^*$ linear dual

Lemma: There's a canonical iso. $(T_pM)^{\text{def. 2}} \cong (T_pM)^{\text{def. 3}}$.

Sketch:

Fact 1 (HW 1, 4d): V vector space $W \subseteq V$ subspace, $\mathrm{Ann}(W) \subseteq V^*$.

$$\mathrm{Ann}(W) = \{\phi \in V^* \mid \phi|_W \equiv 0\}.$$

Then $\mathrm{Ann}(W) \cong (V/W)^*$.

$\phi := \bar\phi \circ \pi \longleftarrow \bar\phi$

$\pi : V \to V/W$

Fact 2: For a derivation $X : C^\infty(p) \to \mathbb{R}$, i.e., $X \in (T_pM)^{\text{def. 2}}$.

Note that any $[f] \in C^\infty(p)$ can be written as const. $+ [\bar f]$, $[\bar f] \in \mathcal{F}_p$. & $X(\text{const.}) = 0$.

$$\Rightarrow \mathrm{Der}(C^\infty(p), \mathbb{R}) \xrightarrow{\cong} \mathrm{Der}(\mathcal{F}_p, \mathbb{R})$$
$$X \longmapsto X|_{\mathcal{F}_p}.$$

Next, note $(X|_{\mathcal{F}_p}) \in \mathrm{Ann}(\mathcal{F}_p^2) \subseteq \mathcal{F}_p^*$.

(calculation: for any derivation X on $C^\infty(p)$, if $f_1, f_2 \in \mathcal{F}_p$, $g \in C^\infty(p)$, then $X(g \cdot f_1 \cdot f_2) = 0$ by leibniz rule.

$$\Rightarrow X|_{\mathcal{F}_p} = \pi^* \bar X|_{\mathcal{F}_p},\quad \bar X|_{\mathcal{F}_p} \in \left(\mathcal{F}_p / \mathcal{F}^2\right)^*.$$

Get a map $(T_pM)^{\text{def } 2} \longrightarrow (T_pM)^{\text{def-}3}$

$X \longmapsto \overline{X}|_{\mathcal{F}_p}$.

<u>Exercise</u>: (e.g., use Taylor's theorem from last class!) This map is also iso.

e.g., note from Taylor's theorem that $\mathcal{F}_p/\mathcal{F}_p^2$, hence $(\mathcal{F}_p/\mathcal{F}_p^2)^*$, has dimension $= m = \dim(M)$.

$\boxtimes$

What we wanted T_pM's for: as a canonical (chart-independent) sources/targets for the derivative of a function at a point.

Given a C^∞ map $f: M^m \longrightarrow N^n$ between smooth manifolds, we can now take its derivative at a point p, to obtain a map

$$df_p := df(p) : T_pM \longrightarrow T_{f(p)}N$$

"derivative of f at p"

"tangent mapping at p"

sometimes "T_pf"

<u>Constructions</u>:

- via def'n 1, $T_pM := C_p/\sim$:

  If $\alpha: I \longrightarrow M$, $0 \longmapsto p$, then define $df_p([\alpha]) = [f\circ\alpha] \in C_{f(p)}/\sim$

  ($f\circ\alpha : I \longrightarrow N$, $0 \longmapsto f(p)$)

  (need to check well-defined).

- via <u>def'n</u> 2: If $X \in T_pM = \mathrm{Der}(C^\infty(p), \mathbb{R})$, then

  define $df_p(X) \in T_{f(p)}N = \mathrm{Der}(C^\infty(f(p)), \mathbb{R})$

  via: $df_p(X)([g]) := X([g\circ f])$

  shorthand for $[(g, U)]$, $U \ni f(p)$ — shorthand for $[(g\circ f, f^{-1}(U))]$, $f^{-1}(U) \ni p$

  $g\circ f : f^{-1}(U) \to U \to \mathbb{R}$.

The main point here is that f induces an algebra map

$$f^*: C^\infty(f(p)) \longrightarrow C^\infty(p) \qquad (*)$$
$$[(g,U)] \longmapsto [(g\circ f, f^{-1}(U))]$$

& hence a map $(-)\circ f^*: \mathrm{Der}(C^\infty(p), \mathbb{R}) \longrightarrow \mathrm{Der}(C^\infty(f(p)), \mathbb{R})$.

- via <u>def'n</u> 3: use $(*)$ to produce a map

$$\mathcal{F}_{f(p)}/\mathcal{F}_{f(p)}^2 \longrightarrow \mathcal{F}_p/\mathcal{F}_p^2 \text{ . , now dualize..}$$

<u>Prop</u>: [Chain rule]: Given $M^m \xrightarrow{f} N^n \xrightarrow{g} Q^q$ C^∞ maps between C^∞ manifolds, $p\in M$, then $d(g\circ f)_p = dg_{f(p)}\circ df_p$.

$$T_pM \xrightarrow{df_p} T_{f(p)}N \xrightarrow{dg_{f(p)}} T_{g(f(p))}Q \text{ .}$$

<u>Lemma</u>: Say $\tilde{U} \underset{\text{open}}{\subseteq} \mathbb{R}^m$, $\tilde{V} \subseteq \mathbb{R}^n$, and $\tilde{f}: \underset{\tilde{p}}{\tilde{U}} \to \tilde{V}$ C^∞ map.

Then with respect to the isomorphism $T_{\tilde{p}}\tilde{U} \cong \mathbb{R}^m$, $\mathrm{Der}(C^\infty(\tilde{p})) \ni \frac{\partial}{\partial x_i} \longleftarrow \vec{e}_i$, the derivative $d\tilde{f}_p$ as defined above becomes the usual derivative map (as we first defined).

$$\begin{array}{ccc} d\tilde{f}_p : T_{\tilde{p}}\tilde{U} & \longrightarrow & T_{\tilde{f}(\tilde{p})}\tilde{V} \\ \wr\Vert & & \wr\Vert \\ \mathbb{R}^m & \dashrightarrow & \mathbb{R}^n \end{array}$$

<u>Cor</u> (of this and chain rule): Given $f: \underset{p}{M} \to N$ smooth, & charts (U,ϕ) of p & (V,ψ) of $f(p)$ with $f(U)\subseteq V$, the usual derivative

$$d(\psi\circ f\circ \phi^{-1})_{\phi(p)}: \mathbb{R}^m \to \mathbb{R}^n$$

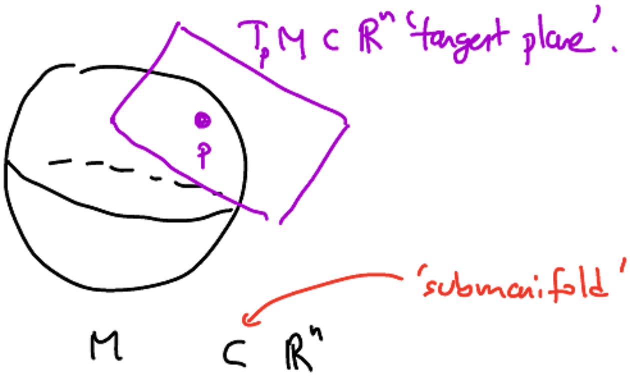
coincides with the composition $d\psi_{f(p)}\circ df_p \circ d(\phi^{-1})_{\phi(p)} : T_{\phi(p)}\phi(U) \to T_{\psi(p)}\psi(V)$

(up to the identification of domain & codomain w/ $\mathbb{R}^m$ & $\mathbb{R}^n$ respectively as in prev. lemma).

(in particular, $\psi: V \to \psi(V)$ & $\phi: U \to \phi(U)$ are smooth maps & can be differentiated (w.r.t. differentiable structures on M, N, Euclidean space))

$V \underset{\text{open}}{\subset} N$, $\psi(V) \underset{\text{open}}{\subset} \mathbb{R}^n$, $U \underset{\text{open}}{\subset} M$, $\phi(U) \underset{\text{open}}{\subset} \mathbb{R}^m$

intuitive picture (to be justified):



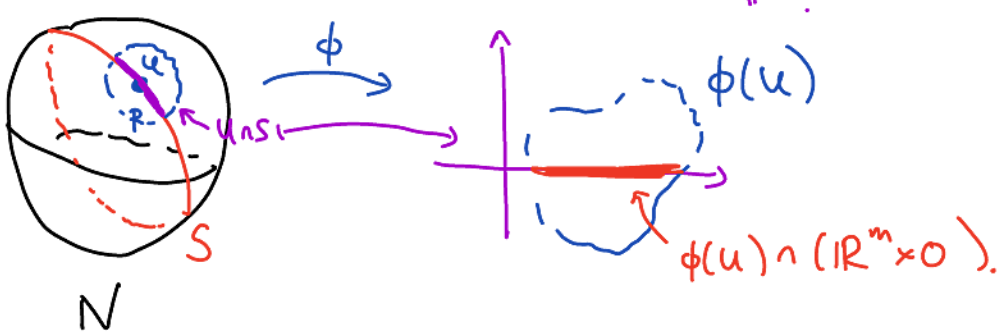
An interlude about submanifolds

First def'n: A subset $S \subseteq N^n$ is an (smooth) m-dimensional submanifold of N if at any point $p \in S$, there exists a chart (U, ϕ) in N's maximal atlas such that

$$\phi(U \cap S) = \phi(U) \cap (\underbrace{\mathbb{R}^m \times \{0\}}_{\subset\ \mathbb{R}^m \times \mathbb{R}^{n-m} = \mathbb{R}^n})$$

(might call such a (U, ϕ) an "adapted" chart to S at p.)

picture:



The pairs $\{(U \cap S,\ \pi_{\mathbb{R}^m} \circ \phi|_{U \cap S})\}$ for adapted charts as above give an atlas for S. $\Rightarrow$ S is an m-dimensional smooth manifold.

First goal: why is e.g., $S^n \subseteq \mathbb{R}^{n+1}$ a submanifold?

$\quad\;\; \{\sum x_i^2 = 1\} = f^{-1}(1)$, $f: \mathbb{R}^{n+1} \to \mathbb{R}$ $\quad f(x_1, \dots, x_{n+1}) = \sum x_i^2$.

More generally:

Given $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ (or $f: M \longrightarrow N$), when is $f^{-1}(y) \subseteq M$ a submanifold? ($y \in N$)

The condition for this involves the derivative df_p at various points $p \in f^{-1}(y)$.