Recall: M^m, N^n, $f: M \to N$ C^∞ map, $p \in M$,

☆ $\rightsquigarrow$ df_p (or $df(p)$): $\underbrace{T_pM}_{\text{vector space of dim. } m} \longrightarrow \underbrace{T_{f(p)}N}_{\text{vector space of dim. } n}$ linear map

Using charts (which identify $T_pM \cong \mathbb{R}^m$, $T_{f(p)}N \cong \mathbb{R}^n$) this becomes usual derivative $d(\psi \circ f \circ \phi^{-1})_{\phi(p)}$.

We can study this map by studying its rank (we'd previously observed this number can be extracted equivalently from rank, for any choice of chart).

By linear alg: $\mathrm{rank}(df_p) \leq \min(m,n)$.

Some special cases worthy of highlight:

Def: • A map $f: M \to N$ has constant rank r means $\mathrm{rank}(df_p) = r$ for every $p \in M$.

(non-example: $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ has $df(p) = [2p]$ non-constant rank, but constant rank as a map $\mathbb{R}\setminus 0 \to \mathbb{R}$)

• A map $f: M^m \to N^n$ is a submersion if f has constant rank n.

($\iff$ df_p is surjective onto $T_{f(p)}N$ at every $p \in M$)

($\Rightarrow$ $m \geq n$ OR $M = \emptyset$).

• A map $f: M^m \to N^n$ is called an immersion if f has constant rank m.

($\iff$ df_p is injective at every $p \in M$)

($\Rightarrow$ $m \leq n$)

Prototype examples:

• of an immersion: $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ $\quad m \leq n$.

$$(x_1, \dots, x_m) \longmapsto (x_1, \dots, x_m, \underbrace{0, \dots, 0}_{n-m})$$

• of a submersion: $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ $\quad m \geq n$.

$(x_1, \dots, x_m) \longmapsto (x_1, \dots, x_n)$ (projection to first n coords.)

• of a const. rank r map: $r \le \min(m,n)$ $\quad \mathbb{R}^m \longrightarrow \mathbb{R}^n$

$$(x_1,\dots,x_r,x_{r+1},\dots,x_m) \longmapsto (x_1,\dots,x_r,\underbrace{0,\dots,0}_{n-r})$$

The key basic theorem that is useful to analyze such maps is:

Lee Thm. 4.12.

Theorem: ("Rank theorem" or a version of 'implicit function theorem')

Say have $f: M^m \to N^n$ smooth with constant rank r. Then, for each $p \in M$ $\exists$ charts (U,φ) in M (the maximal atlas of M), centered at p and (V,ψ) in N centered at $f(p)$, with $f(U) \subseteq V$, s.t. f has the coord. representative:

$$\hat{f} = \psi \circ f \circ \varphi^{-1} : \underset{\substack{\cap \text{ open} \\ \mathbb{R}^m}}{\overset{0=\varphi(p)}{\overset{\cap}{\varphi(U)}}} \longrightarrow \underset{\substack{\cap \text{ open} \\ \mathbb{R}^n}}{\overset{0=\psi(f(p))}{\overset{\cap}{\psi(V)}}}$$

$$\hat{f}: (x_1,\dots,x_r,x_{r+1},\dots,x_m) \longmapsto (x_1,\dots,x_r,0,\dots,0)$$

(i.e., if f immersion ($r=m$): $\hat{f}: (x_1,\dots,x_m) \longmapsto (x_1,\dots,x_m,\underbrace{0,\dots,0}_{n-m})$)

(if f submersion ($r=n$), $\hat{f}: (x_1,\dots,x_m) \longmapsto (x_1,\dots,x_n)$).

Pf sketch: First, replace M, N by $\mathcal{U} \underset{\text{open}}{\subseteq} \mathbb{R}^m$, $\mathcal{V} \underset{\text{open}}{\subseteq} \mathbb{R}^n$,

& p by $0 \in \mathcal{U}$, $f(p)$ by 0 in $\mathcal{V}$.

(why? choose any charts $(\mathcal{U},\phi)$ ~~at~~ centered at p $(\mathcal{V},\psi)$ at $f(p)$ & replace f by $\psi \circ f \circ \phi^{-1}$. It suffices to show we can find charts around $0 \in \mathcal{U}$ & $0 \in \mathcal{V}$ satisfying hypotheses above (exercise)).

Now, by hypothesis, $df(0)$ has rank r. $\Rightarrow$ the matrix of $df(0)$ has an $r \times r$ submatrix which is invertible. ∴ WLOG (reordering coordinates) assume $\left(\frac{\partial f_i}{\partial x_j}\right)_{i=1,\, j=1}^{r \quad r}$ is invertible.

Writing $\mathbb{R}^n = \underset{(v\ ,\ w)}{\mathbb{R}^r \times \mathbb{R}^{n-r}}$, and $\mathbb{R}^m = \underset{(x\ ,\ y)}{\mathbb{R}^r \times \mathbb{R}^{m-r}}$

$v_1,\dots,v_r$ $\quad w_1,\dots,w_{n-r}$ $\qquad x_1,\dots,x_r$ $\quad y_1,\dots,y_{m-r}$.

$\Rightarrow$ write $f(x,y) = (\underset{\mathbb{R}^r}{Q(x,y)}, \underset{\mathbb{R}^{n-r}}{R(x,y)})$

$\Rightarrow \left(\frac{\partial Q^i}{\partial x_j}\right)_{i,j=1}^{r}$ is invertible at $(x,y)=(0,0)$.

Define $\varphi : \mathcal{U} \longrightarrow \mathbb{R}^m$ by $\varphi(x,y) = (Q(x,y), y)$. (so $\varphi(0,0) = (0,0)$).

compute
$$D\varphi(0,0) = \begin{pmatrix} \frac{\partial Q^i}{\partial x_j}(0,0) & \frac{\partial Q_i}{\partial y_j}(0,0) \\ 0 & \underbrace{\delta_{ij} = \frac{\partial y_i}{\partial y_j}}_{\text{Identity}} \end{pmatrix}.$$
is invertible.

Inverse fn. theorem $\Rightarrow \exists\ \mathcal{U}_0 \ni (0,0)$, $\widetilde{\mathcal{U}}_0 \ni \varphi(0,0) = (0,0)$

s.t. $\varphi : \mathcal{U}_0 \xrightarrow{\cong} \widetilde{\mathcal{U}}_0$. $\qquad \widetilde{\mathcal{U}}_0 \underset{\text{open}}{\subseteq} \mathbb{R}^m$

Shrinking if needed, can assume $\widetilde{\mathcal{U}}_0$ is an open cube around 0.

So $\exists$ smooth two-sided inverse

$$\varphi^{-1} : (x,y) \longmapsto (A(x,y), y).$$ (check b/c φ^{-1} inverse to)

for some $A : \widetilde{\mathcal{U}}_0 \to \mathbb{R}^r$.

$\varphi \circ \varphi^{-1} = id \Rightarrow \underline{Q(A(x,y), y) = x}$.

$\Rightarrow f \circ \varphi^{-1}(x,y) = (x, \widetilde{R}(x,y))$ where $\widetilde{R} : \widetilde{\mathcal{U}}_0 \to \mathbb{R}^{n-r}$ second comp. of f.

$\widetilde{R} : (x,y) \longmapsto R(A(x,y), y)$.

$$d(f\circ\varphi^{-1})(x,y) = \begin{pmatrix} \delta_{ij} & 0 \\ \frac{\partial \widetilde{R}_i}{\partial x_j}(x,y) & \frac{\partial \widetilde{R}_i}{\partial y_j}(x,y) \end{pmatrix}.$$

Since φ^{-1} is a diffeo, $d(f\circ\varphi^{-1})$ should have constant rank r at every point in

$\mathcal{U}_0$ by hypothesis

$\Rightarrow$ (b/c first r colums are linearly indep.) $\Rightarrow$ $\dfrac{\partial \widetilde{R}_i}{\partial y_j} \equiv 0$.

$\Rightarrow \widetilde{R}$ is indep. of y.

so set $S(x) := \widetilde{R}(x,0) = \widetilde{R}(x,y)$ any y.

$\Rightarrow f\circ\varphi^{-1}(x,y) = (x, S(x))$

Last step: compose with diffeo. that sends $(x, S(x))$ to $(x,0)$.