Last time:

Theorem: ("Rank theorem" or a version of 'implicit function theorem')

Say have $f: M^m \to N^n$ smooth with constant rank r. Then, for each $p \in M$ $\exists$ charts (U, φ) in (the maximal atlas of M) M centered at p and (V, ψ) in N centered at $f(p)$, with $f(U) \subseteq V$, s.t. f has the coord. representative:

$$\hat{f} = \psi \circ f \circ \varphi^{-1} : \underset{\substack{\cap \text{ open} \\ \mathbb{R}^m}}{\overset{0=\varphi(p)}{\overset{\cap}{\varphi(U)}}} \longrightarrow \underset{\substack{\cap \text{ open} \\ \mathbb{R}^n.}}{\overset{0=\psi(f(p))}{\overset{\cap}{\psi(V)}}}$$

$$\hat{f} : (x_1, \ldots, x_r, x_{r+1}, \ldots, x_m) \longmapsto (x_1, \ldots, x_r, 0, \ldots, 0)$$

Last time:

- reduced to $M = \mathcal{U} =$ (open subset of) $\mathbb{R}^m$, $N = \mathcal{V} =$ (open subset of) $\mathbb{R}^n$, $p = 0$, $f(p) = 0$.
- using the notation $\mathbb{R}^m = \underset{(x,\ y)}{\mathbb{R}^r \times \mathbb{R}^{m-r}}$, $\mathbb{R}^n = \underset{(v,\ w)}{\mathbb{R}^r \times \mathbb{R}^{n-r}}$,

we found a smaller $\underset{0}{\overset{\psi}{U_0}} \underset{\text{open}}{\subseteq} \mathcal{U}$, $\tilde{U}_0$ open cube in $\mathbb{R}^m$, diffeomorphism $\varphi : U_0 \xrightarrow{\cong} \tilde{U}_0$ s.t.

$$(*) \quad f \circ \varphi^{-1} : (x, y) \longmapsto (x, S(x)) \text{ for some } S : \mathbb{R}^r \to \mathbb{R}^{n-r}.$$

Let $\underset{0}{\overset{\psi}{V_0}} \underset{\text{open}}{\subseteq} V$, $V_0 = \{ (v, w) \in \mathcal{V} \mid (v, 0) \in \tilde{U}_0 \}$.

$\tilde{U}_0$ a cube $\underset{\text{by } (*)}{\Rightarrow} f \circ \varphi^{-1}(\tilde{U}_0) \subseteq V_0$, i.e., $f(U_0) \subseteq V_0$.

Define $\psi : V_0 \to \mathbb{R}^n$ by $\psi(v, w) = (v, w - S(v))$.

- diffeomorphism onto its image (exercise: write inverse).
- $\psi \circ f \circ \varphi^{-1} : (x, y) \longmapsto (x, 0)$ as desired). $\square$

Thm: $f: M^m \to N^n$ a (smooth) submersion (for every $p \in M$, $df_p : T_pM \to T_{f(p)}N$ has rank n);

($\Rightarrow$ if $M \neq \phi$, then $m \geq n$)

& pick $y \in N$ any point. Then $f^{-1}(y) \subseteq M$ is a submanifold of dimension $m-n$.

(Note: get same outcome if only require f to be a submersion on an open set $\mathcal{U} \supseteq f^{-1}(y)$, by restricting f to $f|_{\mathcal{U}} : \mathcal{U} \to N$, seeing as $\mathcal{U}$ is a smooth manifold).

Pf of theorem: Denote by $S = f^{-1}(y)$, and let $p \in S$. By constant rank theorem, $\exists$ charts (U, ϕ) centered at p & (V, ψ) centered at $y = f(p)$, with $f(U) \subseteq V$, s.t.

$$\hat{f} = \psi \circ f \circ \phi^{-1} : \underset{\substack{\cap\text{ open} \\ \mathbb{R}^m}}{\underset{\phi(p)=0}{\phi(U)}} \longrightarrow \underset{\substack{\cap\text{ open} \\ \mathbb{R}^n}}{\underset{\psi(y)=0}{\psi(V)}}$$

is the "model submersion" $(x_1, \ldots, x_m) \longmapsto (x_1, \ldots, x_n)$.

Note that $\phi(\underset{f^{-1}(y)}{S} \cap U) = \hat{f}^{-1}(0) \cap \phi(U) = (\{0\} \times \mathbb{R}^{m-n}) \cap \phi(U)$

as desired, so (U, ϕ) is a chart adapted to S around $p \in S$. ∎

Example: Consider $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ $\quad f(x,y) = x^2 + y^2$.

Not a submersion on all of $\mathbb{R}^2$ b/c at $(0,0)$, $df(0,0) = [2x, 2y]\big|_{(x,y)=(0,0)} = [0,0]$ is not surjective

However, it is a submersion when restricted to

$$\mathcal{U} = \mathbb{R}^2 \setminus (0,0)$$

(as $[2x\ 2y] \neq [0,0]$ when $(x,y) \neq (0,0)$, hence $df(x,y)$ surjective on $\mathcal{U}$).

so $f: \mathbb{R}^2 \setminus \vec{0} \to \mathbb{R}$ is a submersion.

$\Rightarrow f^{-1}(1) \cap \mathbb{R}^2 \setminus 0 \subseteq \mathbb{R}^2 \setminus 0$ submanifold.

Now, since $f^{-1}(1) \subseteq \mathbb{R}^2 \setminus 0$, as $f(0,0) \neq 1$,

$\Rightarrow f^{-1}(1) \subseteq \mathbb{R}^2 \setminus 0 \subseteq \mathbb{R}^2$ submanifold.

$\Rightarrow$ another construction of $S^1 = f^{-1}(1)$ as a manifold.

Prop: Say $p \in M$, $f : M^m \to N^n$ any map which has full rank at p.
(meaning $\operatorname{rank}(df_p) = \min(m,n)$, i.e., f is a submersion or an immersion).
Then, $\exists$ an open set $U \ni p$ s.t. df_x has full rank at every $x \in U$.

Cor: If df_p is surjective for every $p \in f^{-1}(y)$, then $\exists$ an open subset $U \supseteq f^{-1}(y)$ s.t. $f|_U$ is a submersion.

Cor: If y is a *regular value* of $f : M^m \to N^n$, meaning that $\forall\, p \in f^{-1}(y)$, df_p is surjective, then $f^{-1}(y) \subseteq M$ is a submanifold of dimension $m-n$.

Call $x \in N$ a *critical value* of $f : M \to N$ if it's not a regular value, i.e., if $\exists\, p \in f^{-1}(x)$ with df_p not surjective; call such a p a *critical point* of f.

Proof of Prop: Pick a chart around p & $f(p)$ in which f becomes

$$f \rightsquigarrow \hat f := \psi \circ f \circ \phi^{-1} : \underset{\mathbb{R}^m}{\underset{\cap\ \text{open}}{\phi(\overset{\hat p = \phi(p)}{U})}} \longrightarrow \underset{\mathbb{R}^n.}{\underset{\cap\ \text{open}}{\psi(V)}}$$

Know: $d\hat f(\hat p)$ is surjective.

It suffices to show $\exists\ \underset{\hat p}{\hat W} \ni \hat p$ s.t. $d\hat f(x)$ is surjective for every $x \in \hat W$.

(this will imply that df_x surjective $\forall\, x \in W$, $W = \phi^{-1}(\hat W) \underset{\text{open}}{\subseteq} U \subseteq M$).

by smoothness this is continuous

Can think of $x \mapsto d\hat f(x)$ as a map $\underset{\mathbb{R}^m}{\underset{\cap\ \text{open}}{\phi(U)}} \xrightarrow{d\hat f} \text{Mat}(m \times n)$

key: $\boxed{\text{this is an open subset}}$

$\cup\!\!\mid$ Full Rank Mat$(m\times n)$

$\Rightarrow$ by continuity, since $\hat p \in (d\hat f)^{-1}(\text{Full Rank Mat}(m\times n))$,

$\exists\ \hat W \ni \hat p$ on which $d\hat f$ is full rank. $\blacksquare$

Note: It would be helpful to have 'intrinsic' (ind. of specific chart) way to say that $p \longmapsto df_p$ 'continuously/smoothly' varies!

coming soon, c.f. the notion of tangent bundle, cotangent bundle, one-form..

Examples of critical values, critical points, regular values.

Ex: $M = \{x^4+y^4+z^4+w^4=3\} \subseteq \mathbb{R}^4$. Is it a (sub)manifold of dimension 3?

Have a function $f: \mathbb{R}^4 \to \mathbb{R}$ $\quad (x,y,z,w) \longmapsto x^4+y^4+z^4+w^4$, $\quad M = f^{-1}(3)$.

The derivative matrix is $df(\overbrace{x,y,z,w}^{\vec{a}}) = [4x^3, 4y^3, 4z^3, 4w^3]$.

$\text{Rank}(df(\vec{a})) = 1$ unless $\vec{a} = (0,0,0,0)$, in which case $df(0,0,0,0)$. So $\vec{0}$ is the only crit. point of f, & note $\vec{0} \notin f^{-1}(3)$ b/c $f(\vec{0}) = 0$.

So 3 is a regular value of f (b/c $df(x)$ surjective $\forall x \in f^{-1}(3)$).

$\Rightarrow M$ is a (sub)manifold of $\mathbb{R}^4$ of dimension 3.

☆ Check: $T_pM = \ker(df_p) \subseteq T_p\mathbb{R}^4 = \mathbb{R}^4$.

Involved ex:

Let $SL_n(\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) \mid \det(A) = 1\}$ special linear group of $n\times n$ real matrices

Claim: this is a submanifold of $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

write $SL_n(\mathbb{R}) = f^{-1}(1)$ where

$$f: M_{n\times n}(\mathbb{R}) \xrightarrow[\det]{} \mathbb{R}.$$

Main claim: 1 is a regular value of f. $\Rightarrow SL_n(\mathbb{R})$ is a manifold of dim. n^2-1.