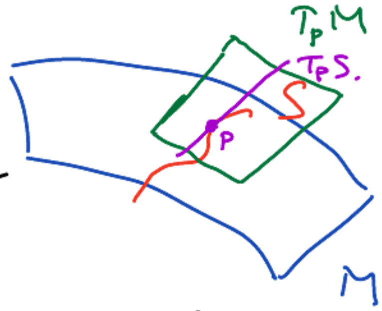
$S \subseteq M$ submanifold, $p \in S$ a point. Then there's a natural inclusion

$$T_pS \hookrightarrow T_pM$$



(How to see this? exercise: try to see w/ all 3 def'ns of tangent space. Def'n 2:

$$T_pS = \mathrm{Der}(C^\infty(p\in S), \mathbb{R}) \hookrightarrow T_pM = \mathrm{Der}(C^\infty(p \in M), \mathbb{R})$$

$$X \longmapsto X \circ (\text{restriction}: C^\infty(p\in M) \to C^\infty(p\in S)).$$

$$[f] \longmapsto [f|_S].$$

To check (exercise/lemma): (1) Above map is injective

(2) If $S = f^{-1}(y)$, $f: M \to N$, $y \in N$ regular value, then at $p \in S = f^{-1}(y)$,

can check $T_pS = \ker(df_p : T_pM \to T_yN)$

(can check this in an 'adapted' chart near p (using constant rank theorem) in which, after applying chart diffeos, f, M, N, S become:

$$\underbrace{\{0\}\times\mathbb{R}^{m-n}}_{\hat{S} = \hat{f}^{-1}(0)} \subseteq \underbrace{\overset{\hat{p}}{\mathbb{R}^m}}_{\hat{M}} \xrightarrow{\hat{f}} \underbrace{\overset{\hat{y}=0}{\mathbb{R}^n}}_{\hat{N}} \quad)$$

$$(x_1, \dots, x_m) \longmapsto (x_1, \dots, x_n)$$

recall that in this chart: $d\hat{f}_p : \mathrm{Der}(C^\infty(p\in\hat{M}), \mathbb{R}) \to \mathrm{Der}(C^\infty(\hat{f}(p)\in\hat{N}), \mathbb{R})$

$$X \longmapsto d\hat{f}(X)([g]) = X([g\circ\hat{f}])$$

If $X = Y \circ (\text{restriction}: C^\infty(p\in\hat{M}) \to C^\infty(p\in\hat{S}))$

then $df(X)[g] = Y \circ \text{restriction}([g\circ\hat{f}])$

$= Y([g\circ\hat{f}]|_S) = Y(0) = 0.$

(b/c $\hat{f}|_S \equiv 0$).

We've proven $T_pS \subseteq \ker df_p$, exercise to prove reverse inclusion.

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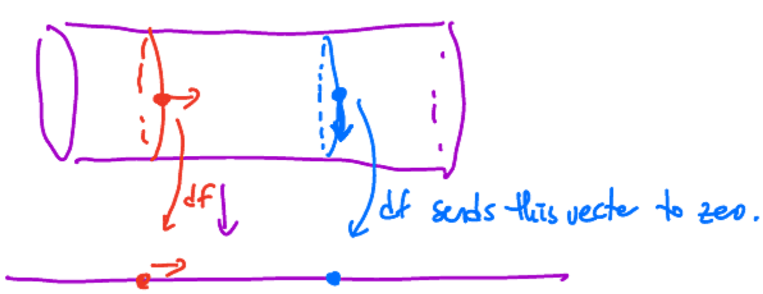
picture of a submersion:



Immersions and Embeddings

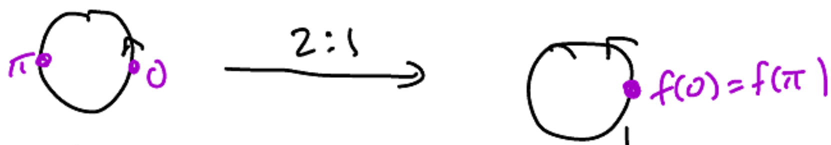
Recall $f: M^m \to N^n$ is an *immersion* if $\forall\, p \in M$, df_p is injective
($\Rightarrow$ if $M \neq \emptyset$ that $n \geq m$)

Examples of immersions:

(1)

$$S^1 \overset{f}{\hookrightarrow} \mathbb{R}^2$$

(1.5)



$$\begin{array}{ccc} S^1 & & S^1 \\ \cap & & \cap \\ \mathbb{C} & \xrightarrow{z^2} & \mathbb{C} \end{array}$$

(2) $F: \mathbb{R} \to S^1 \subset \mathbb{C}$
$\quad t \longmapsto e^{it}$

(3) $f: \mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2 \qquad t \longmapsto [(at, bt)]$, where b/a is irrational.
f is injective
& an immersion, w/ dense image in $\mathbb{R}^2/\mathbb{Z}^2$.
(exercise: image of f is not a submanifold)

Recall some point-set topology terminology: X top. space

- $V \subset X$ is closed if $X \setminus V$ open.
- $A \subset X$ any set, the closure of A is $\overline{A} := \bigcap\limits_{\substack{S \subseteq X \text{ closed} \\ A \subseteq S}} S$.
- A subset $V \subset X$ is *compact* if any open cover of V has a finite subcover.
  - a metric space is cpct iff any sequence has a convergent subsequence

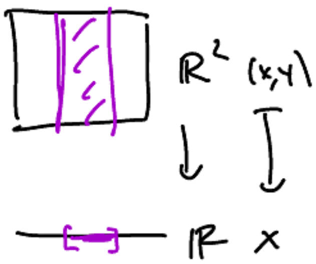
- $A \subseteq \mathbb{R}^n$ is cpct. iff it is closed and bounded

(means $A \subseteq B_R(x_0)$, some x_0, R).

- $f: X \to Y$ is *proper* if whenever $V \subseteq Y$, $f^{-1}(V)$ is compact too.

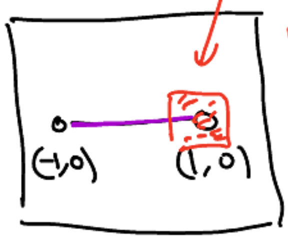
(note that for any continuous f, if $K \subseteq X$ cpct, $f(K) \subseteq Y$ is cpct.)

non-ex: projection



- $\mathbb{R} \to \mathbb{R}^2$, $t \mapsto \left(\frac{t}{|t|+1}, 0\right)$ has image.

cpct this K has non-compact preimage



Embeddings and submanifolds (revisited)

Def: An *embedding* $M^m \xrightarrow{f} N^n$ (all C^∞) is an immersion which is one-to-one and proper.

(sometimes proper isn't part of the def'n, & the above def'n is called a "proper embedding").

Note: None of the examples of immersions above were embeddings. (why?)

Def 2 of submanifold:

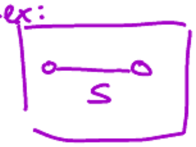
A *submanifold* S of N^n of dimension m is the image of an embedding

$$S = f(M^m), \quad f: M^m \to N^n$$

(immersion, one-to-one, proper).

Lemma: This def'n is equivalent to Def 1 of submanifold (given in terms of $\exists$ adapted charts), assuming further in Def 1 that the inclusion $S \subseteq M$ is proper.

(non-ex:



$M = \mathbb{R}^2$

Pf: Exercise (HW).

Prop: M^m, N^n manifolds w/ topologies $\mathcal{T}_M, \mathcal{T}_N$. If $f: M^m \hookrightarrow N^n$ is an embedding, then $f^{-1}(\mathcal{T}_N) = \mathcal{T}_M$ (meaning the topology on M is the one induced by topology on N).

Pf: $f^{-1}(\mathcal{T}_N) \subseteq \mathcal{T}_M$ b/c f is continuous. Exercise: show $\mathcal{T}_M \subseteq f^{-1}\mathcal{T}_N$.

Notes: general questions we could ask:

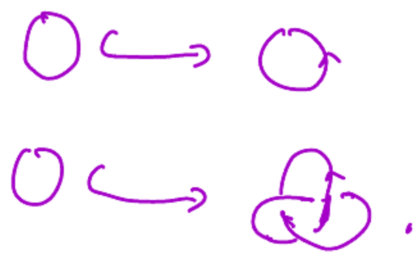
- when does there exist $M^m \xhookrightarrow{f} N^n$?
- when is $f: M^m \hookrightarrow N^n$ "unique" ?

↑ need to clarify what we mean here, by introducing an equivalence relation & saying "unique up to equivalence"

e.g., <u>isotopy</u> is $f_t : M \hookrightarrow N$, $t \in [0,1]$ where $(t, m) \mapsto f_t(m)$ smooth (between f_0 & f_1), <u>and each f_t is an embedding</u>

knot theory: $\exists$ non-isotopic embeddings $S^1 \hookrightarrow \mathbb{R}^3$

e.g.,



<u>Question for next time</u>: When does $M \hookrightarrow \mathbb{R}^N$ and for what N's does a given embedding exist?