Last time: When is there an embedding $M^m \hookrightarrow \mathbb{R}^N$?

Today: show one exists for $N >> 0$, then later study question of reducing N.

Technical tool:

Partitions of unity. A collection of smooth functions $\{\psi_\alpha : M \to \mathbb{R}_{\geq 0}\}_{\alpha \in I}$ is a (differentiable) partition of unity if:

(a) $\operatorname{supp} \psi_\alpha$ is compact for all α.

(b) The collection of supports $\{\operatorname{supp}(\psi_\alpha) = \overline{\{x \in M \mid \psi_\alpha(x) \neq 0\}}\}_{\alpha \in I}$ is locally finite (meaning for every $p \in M$, some neighborhood $U \ni p$ intersects only finitely many supports).

(c) $\psi_\alpha(p) \geq 0$ for every p, α, and

$$\sum_{\alpha \in I} \psi_\alpha(p) = 1 \quad \text{for every } p \quad \left(\text{i.e., } \sum_{\alpha \in I} \psi_\alpha \equiv 1\right).$$

by (b) only finitely many non-zero $\psi_\alpha(p)$'s for any p, so sum makes sense, and is smooth.

If $\{U_\beta\}_{\beta \in J}$ is an open cover of M, we say $\{\psi_\alpha\}_{\alpha \in I}$ is subordinate to $\{U_\beta\}$ if for every α $\exists$ β $\operatorname{supp} \psi_\alpha \subseteq U_\beta$.

& say $\{\psi_\alpha\}_{\alpha \in I}$ subordinate to $\{U_\alpha\}_{\alpha \in I}$ with same index if $\operatorname{supp} \psi_\alpha \subset U_\alpha$ for all $\alpha \in I$.

Prop: If $\{U_\alpha\}$ any open cover of a manifold M, then $\exists$ a partition of 1 subordinate to U_α.

Proof: in stages: (we'll prove in special case M is compact for simplicity).

Step 1: Consider $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$

easy: f is C^∞.

Step 2: Take $g_{ab} : \mathbb{R} \to \mathbb{R}$ as $g_{ab}(x) = f(x-a) \cdot f(b-x)$ $\quad$ ($a < b$ real #'s)

$\Rightarrow$ g is a bump function, meaning $\bullet\, g \geq 0$, $\bullet\, \operatorname{supp}(g) = [a,b]$, $\bullet\, g > 0$ on (a,b).

Step 3: Consider a bump function supported on

$(*) = [a_1, b_1] \times \cdots \times [a_m, b_m] \subset \mathbb{R}^m_{x_1, \ldots, x_m}$.

defined by $\psi(\vec{x}) = g_{a_1 b_1}(x_1) \cdot g_{a_2 b_2}(x_2) \cdot \ldots \cdot g_{a_m b_m}(x_m)$,

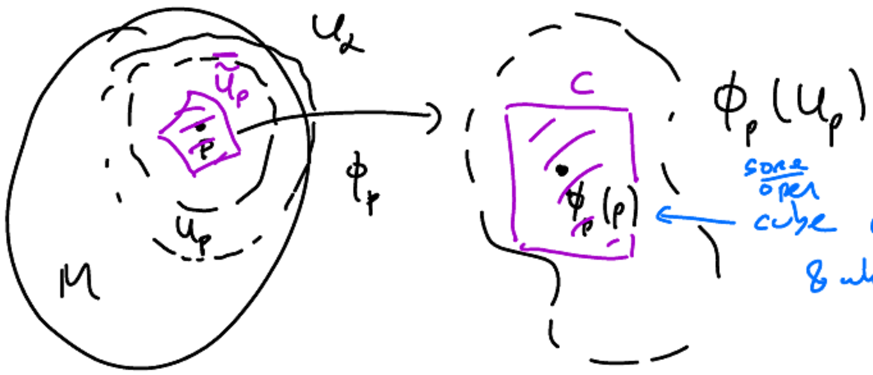
$\Rightarrow$ ψ supported on $(*)$, positive on interior.

Step 4: (assume M is compact for simplicity).

For each $p \in M$, choose a chart (U_p, ϕ_p) around p contained in some U_α, and let $\tilde{U}_p$ be preimage of (interior of) a cube C containing $\phi_p(p)$ in $\phi_p(U_p)$.

$\Rightarrow$ Get a function ψ_p on U_p with support in $\overline{\tilde{U}_p} \subseteq U_p$ $\quad$ ($\psi_p := \underset{\uparrow}{\psi} \circ \phi_p$).

bump function for C.



some open cube whose closure lies in $\phi_p(U_p)$ & which contains $\phi_p(p)$.

observe: ψ_p extends by 0 to a smooth function $\psi_p : M \to \mathbb{R}$.

& $\operatorname{supp} \psi_p \subseteq$ some U_α.

The $\{\tilde{U}_p\}_{p \in M}$ cover M, so $\exists$ a finite subcover $\tilde{U}_{p_1}, \ldots, \tilde{U}_{p_k}$ $\boxed{\text{by compactness.}}$

Consider the functions $\{\psi_{p_i}\}_{i=1}^k$. Note that $\sum_{i=1}^k \psi_{p_i} > 0$ at every point p.

Now, let $\tilde{\psi}_{p_i} = \dfrac{\psi_{p_i}}{\sum_{j=1}^k \psi_{p_j}}$. Then $\sum \tilde{\psi}_{p_i} = 1$. So $\{\tilde{\psi}_{p_i}\}_{i=1}^k$ is a partition of unity subordinate to $\{U_\alpha\}$.

Application I:

Thm: If M is compact of dimension m, $\exists$ a number N and an embedding

$f: M \hookrightarrow \mathbb{R}^N$.

Pf: Let $\{(U_i, \phi_i)\}_{i \in I}$ some atlas for M (in given smooth structure), and $\{\psi_j : M \to \mathbb{R}\}_{j \in J}$ partition of unity subordinate to $\{U_i\}_{i \in I}$.

WLOG b/c M is cpct. assume I, J finite, say $J = \{1, \ldots, k\}$.

by def'n, for each $j \in J$ $\operatorname{supp} \psi_j \subset U_{i_j}$

Now let $N = k + k \cdot m$ and define

$f : M \hookrightarrow \mathbb{R}^N$ by

$$f(x) = (\underbrace{\psi_1(x)}_{\mathbb{R}}, \underbrace{\psi_2(x)}_{\mathbb{R}}, \ldots, \underbrace{\psi_k(x)}_{\mathbb{R}}, \underbrace{\psi_1(x)\phi_{i_1}(x)}_{\mathbb{R}^m}, \underbrace{\psi_2(x)\phi_{i_2}(x)}_{\mathbb{R}^m}, \ldots, \underbrace{\psi_k(x)\phi_{i_k}(x)}_{\mathbb{R}^m})$$

$\phi_{i_1} : U_{i_1} \to \mathbb{R}^m$.

& $\psi_1 \phi_{i_1} : U_{i_1} \to \mathbb{R}^m$. extends smoothly by 0 to all of M. Meaning define $\psi_1 \phi_{i_1}(y) = 0$ if $y \notin U_{i_1}$. (exercise: check).

Claims (1) f is injective:

say $x, y \in M$, $x \neq y$. ; want $f(x) \neq f(y)$

- if $x, y \in$ same U_j, and $\psi_j(x) = \psi_j(y)$ done (b/c $\phi_{i_j}(x) \neq \phi_{i_j}(y)$ for $x \neq y$!) and if $\psi_j(x) \neq \psi_j(y)$ then done too.

- If $x, y \in$ different U_j's, then can find ψ_i, ψ_j with $\psi_i(x) > 0$, $\psi_i(y) = 0$ & $\psi_j(x) = 0$, $\psi_j(y) > 0$.

$\Rightarrow$ done.

(2) f is an immersion. (exercise, reduce to the fact that each $d\phi_i$ is an immersion where defined).

Q: How much can we reduce N?

Thm: (Whitney embedding theorem): $\exists$ an embedding $M^n \hookrightarrow \mathbb{R}^{2n+1}$.

Idea: we'll argue given a fixed $M^n \hookrightarrow \mathbb{R}^N$ if $N > 2n+1$, composing with "most linear projections" $\mathbb{R}^N \to \mathbb{R}^{N-1}$ preserve embedding property.

"Sard's theorem".