Today: • the tangent bundle of a manifold. ← structure which allows us to think of various T_pM as p varies "varying smoothly", & that df_p "varies smoothly in p."

• introduce Sard's theorem + application to Whitney embedding.

Tangent bundle M^m m'dim'l (smooth) manifold.

Let $TM = \coprod_{p\in M} T_pM = \{(p,v) \mid p\in M, v\in T_pM\}$. This is called the tangent bundle of M, and it comes with a natural projection map

$$\pi: TM \longrightarrow M$$
$$(p,v) \longmapsto p.$$

Each T_pM is called the "fibre" of this projection map at the point p.

We topologize TM so that it's a topological space & in fact a C^∞ manifold of dimension $2m$, with π a C^∞ map.

Given $\mathcal{A} = \{(U_i, \phi_i)\}_{i\in I}$ an atlas for M, so $\phi_i : U_i \xrightarrow{\sim} \underset{\phi_i(U_i)}{V_i} \underset{\text{open}}{\subset} \mathbb{R}^m$,

for each U_i set $\widetilde{U}_i := \pi^{-1}(U_i) \subset TM$, $\widetilde{U}_i = \{(p,v) \mid p\in U_i\}$, and define

$$\widetilde{\phi}_i : \widetilde{U}_i \longrightarrow V_i \times \mathbb{R}^m \underset{\text{open}}{\subseteq} \mathbb{R}^{2m}$$
$$(p,v) \longmapsto (\phi_i(p), d(\phi_i)_p(\vec{v}))$$

$d(\phi_i)_p : T_pM \longrightarrow T_{\phi_i(p)}V_i \underset{\text{canon.}}{\cong} \mathbb{R}^m$.

(exercise: bijection).

We'll obtain a topology on TM by declaring $\widetilde{\phi}_i$ to be a homeomorphism (i.e., in $\widetilde{U}_i$, $w\subset\widetilde{U}_i$ open iff $\widetilde{\phi}_i(w)$ is open), and that $\{\widetilde{U}_i\}$ are an open cover for TM.

($\Longleftrightarrow$ $U\subseteq TM$ is open iff $\widetilde{\phi}_i(U\cap\widetilde{U}_i)$ is open in $\mathbb{R}^{2m}$ $\forall\, i\in I$).

note: topology on TM only depends on $[\mathcal{A}]$ or $\mathcal{A}_{max}$ on M, and:

Lemma: (w.r.t. above topology), the collection $\{(\widetilde{U}_i, \widetilde{\phi}_i)\}_{i\in I}$ is a differentiable atlas for TM, making TM a smooth $2m$-dim'l manifold. (& moreover resulting diff. structure on TM only depends on differentiable structure on M).

Pf: Note first that $\bigcup \widetilde{U}_i = TM$ & each $\widetilde{U}_i$ is open.

Exercise: check TM w/ topology above is Hausdorff, second countable.

<u>Transition maps</u>: We need to check that if $(\widetilde{U}_i, \widetilde{\phi}_i)$ $(\widetilde{U}_j, \widetilde{\phi}_j)$ are two charts with $\widetilde{U}_i \cap \widetilde{U}_j \neq \emptyset$ (so $U_i \cap U_j \neq \emptyset$), the transition maps:

$$\begin{array}{ccccc} \widetilde{\phi}_j(\widetilde{U}_i \cap \widetilde{U}_j) & \xrightarrow{\widetilde{\phi}_j^{-1}} & \widetilde{U}_i \cap \widetilde{U}_j & \xrightarrow{\widetilde{\phi}_i} & \widetilde{\phi}_i(\widetilde{U}_i \cap \widetilde{U}_j) \\ \cap\!\shortmid & & & & \cap\!\shortmid \\ \mathbb{R}^m \times \mathbb{R}^m & & & & \mathbb{R}^m \times \mathbb{R}^m \end{array}$$

$$(p, v) \longmapsto (\phi_i(p), d(\phi_i)_p(\vec{v}))$$

$$(q, w) \longmapsto (\phi_j^{-1}(q), d(\phi_j^{-1})_q(w)) \longmapsto (\phi_i \phi_j^{-1}(q), d(\phi_i)_{\phi_j^{-1}(q)} d(\phi_j^{-1})_q w)$$

$$\| \text{ chain rule}$$

$$(q,w) \longmapsto (\phi_i \circ \phi_j^{-1}(q), d(\phi_i \circ \phi_j^{-1})_q w).$$

are smooth. This follows from the above computation & smoothness of $\phi_i \circ \phi_j^{-1}$. (why?).

$\boxtimes$.

let $f = \phi_i \circ \phi_j^{-1}$.

study $(p, w) \longmapsto df_p(w)$

$\mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$.

each is smooth in p and w.

$$(p, w = (w_1, \dots, w_m)) \longmapsto \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \\ \vdots & \ddots & \\ & & \frac{\partial f_m}{\partial x_m}(p) \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}.$$

<u>Lemma</u>: $\pi : TM \to M$ is a smooth map.

<u>Examples of tangent bundles</u>:

- $U \underset{\text{open}}{\subseteq} \mathbb{R}^m$. Then $TU \underset{\text{diff.}}{\cong} U \times \mathbb{R}^m \underset{\text{open}}{\subseteq} \mathbb{R}^m \times \mathbb{R}^m$

  A tangent vector is a pair (q, v), $q \in U$, $\vec{v}$ a tangent direction in $\mathbb{R}^m$; frequently written as $\vec{v} = \sum a_i \frac{\partial}{\partial x_i}$. (this uses $T_q U \cong \mathrm{Der}(C^\infty(q), \mathbb{R})$

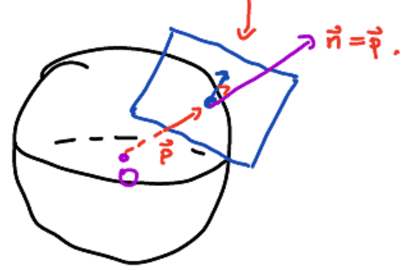
  $$\begin{array}{ccc} \mathbb{R}^m & \longrightarrow & \frac{\partial}{\partial x_i}. \\ \vec{e}_i & \longmapsto & \end{array}$$

- $S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\} \subseteq \mathbb{R}^3$.

  Calculus-style definition of TS^2: think of $TS^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$

$\vec{n}$ & tangent vector are orthogonal.

consisting of coordinates $(\vec{p}, \vec{v})$ where $\vec{p} \in S^2$ i.e. $\|\vec{p}\| = 1$ and $\vec{v} \perp \vec{p}$.



e.g., $TS^2 \underset{\text{diff.}}{\overset{?}{\cong}} \{(p,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|p\| = 1, \ \vec{v} \cdot \vec{p} = 0\}$. ← on HW.

• More generally (also on HW): $f: \mathbb{R}^{m+k} \to \mathbb{R}^k$, $y \in \mathbb{R}^k$ regular value of f, & $M = f^{-1}(y)$, then check that

$$TM \underset{\text{diff.}}{\cong} \left\{ (p,v) \in \mathbb{R}^{m+k} \times \mathbb{R}^{m+k} \;\middle|\; \begin{array}{l} p \in M = f^{-1}(y), \\ df_p(v) = 0 \text{ so } v \in \ker df_p \end{array} \right\}.$$

(recovers above example for $f = x_1^2 + x_2^2 + x_3^2$ b/c

$$df_p(\vec{v} = (v_1, v_2, v_3)) = 2\vec{p} \cdot \vec{v} = 0 \text{ iff } \vec{p} \cdot \vec{v} = 0).$$
$$\| \qquad\qquad$$
$$[2p_1 \ 2p_2 \ 2p_3] \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

Sard's theorem:

$f: M^m \to N^n$ smooth map, for $y \in N$, say:

- y is a regular value if for every $p \in f^{-1}(y)$, $df_p: T_pM \to T_{f(p)}N$ is surjective.
  ($\Rightarrow$ if $f^{-1}(y)$ non-empty then $m \geq n$),
- y is a crit. value if it's not a regular value. (i.e., if $\exists$ $p \in f^{-1}(y)$ w/ df_p not surjective.)
  ↖ called a critical point.

Importance of this concept was that for a regular value, $f^{-1}(y) \subseteq M$ is, if nonempty, a submanifold of dimension $m-n$. (smooth)

Q: can we find regular values for a given f? for any f?

Sard's theorem: there are "many" regular values. More precisely:

Thm: [Sard]: $f: M \to N$ any smooth map. Then the subset of N consisting of critical values has measure zero in N. ($\Rightarrow$ the set of regular values has "full measure," in particular is dense in N. In particular, $\exists$ reg. value of f in any open set in N).

Ex: $f: \mathbb{R} \to \mathbb{R}$ constant. Then 0 is critical value, any $x \neq 0$ is a regular value & $f^{-1}(x)$ is empty.

Next time: Define measure zero, prove Sard, apply to embedding results.