Last time, stated:

(i.e., image of all critical points)

Thm: [Sard]: $f: M^m \to N^n$ any smooth map. Then the subset of N consisting of critical values has measure zero in N. ($\Rightarrow$ the set of regular values has "full measure," in particular is dense in N. In particular, $\exists$ reg. value of f in any open set in N).

Note: If $m = \dim(M) < n = \dim(N)$, then every $p \in M$ is a critical point of a given $f: M \to N$.
$\Rightarrow$ critical values of f are $f(M)$. Sard's theorem $\Rightarrow$ the image $f(M) \subseteq N$ has measure zero $\Rightarrow$ So f cannot be surjective, and complement $N \setminus f(M)$ is dense, i.e., "many" points $y \in N$ with $f^{-1}(y) = \emptyset$.

Def: A subset $A \subseteq \mathbb{R}^n$ has measure 0 if for every $\varepsilon > 0$ $\exists$ countably many boxes $I_1, I_2, \ldots, I_k, \ldots$; $I_s = [a_1^s, b_1^s] \times \cdots \times [a_N^s, b_N^s]$ s.t. $A \subset \bigcup_{s \in \mathbb{N}} I_s$ and $\sum \mathrm{vol}(I_s) < \varepsilon$.

Rmk: "measure" refers to "Lebesgue measure"

Lemmas: (omitted proof):

(1) Any open $U \subseteq \mathbb{R}^n$ does not have measure zero.

$\Rightarrow$ a measure 0 A cannot contain an open U.

$\Rightarrow$ any open $U \subseteq \mathbb{R}^n$ contains a point not in such an A.

$\Rightarrow \mathbb{R}^n \setminus A$ is dense in $\mathbb{R}^n$ (if A has measure 0).

(2) If $A_1, A_2, \ldots, A_k, \ldots$ countably many subsets of measure zero, then $\bigcup_{i \in \mathbb{N}} A_i$ still has measure 0.

(since any point has measure 0, (2) $\Rightarrow$ any countable subset has measure 0 e.g., $\mathbb{Q}^n \subseteq \mathbb{R}^n$).

(3) Say $A \subseteq \mathbb{R}^n$ cpct. subset and $A \cap \{c\} \times \mathbb{R}^{n-1}$ has measure 0 in $\mathbb{R}^{n-1}$ for every $c \in \mathbb{R}$. Then, A has measure 0 in $\mathbb{R}^n$.

(4) The graph of $f: \mathbb{R}^{n-1} \to \mathbb{R}$, or an affine proper subspace $H \subset \mathbb{R}^n$, all have measure 0 in $\mathbb{R}^n$.

smooth m-m'fold

Def: A subset $A \subset M^m$ has <u>measure 0</u> if $\exists$ atlas $\mathcal{A} = \{(U_i, \phi_i)\}$ in differentiable structure on M such that each $\phi_i(A \cap U_i)$ has measure zero.

Independence of choice of atlas used: exercise, follows from:

Prop: Any diffeo. $f: U \xrightarrow{\cong} V$ preserves the notion of a subset of measure zero.

$U \subset \mathbb{R}^m$ open, $V \subset \mathbb{R}^n$ open

(Pf omitted).

<u>Cor of above Lemma:</u> If $A \subset M^m$ measure zero, then $M \setminus A \subseteq M$ is dense.

<u>Sketch of proof of Sard's theorem:</u>

<u>Statement</u>: $f: M^m \to N^n$ then critical values$(f) \subseteq N$ has measure 0.

Induct on $m = \dim(M)$.

- base case: $m = 0$ (M is countable discrete set). immediate b/c:
  - if $n = 0$ then there are no critical points.
  - if $n > 0$ the set of critical values is $f(M)$, a countable subset of N, therefore measure 0.

- General m, assume results hold for all maps to N from any k-dim'l manifold $k \leq m-1$:

By covering M & N by a <u>countable</u> collection of charts adapted to f, we reduce to the case of

$$F: U \longrightarrow V$$

$U \subset \mathbb{R}^m$ open, $V \subset \mathbb{R}^n$ open.

Let $C :=$ set of critical points of F. Want $F(C)$ has measure 0 in V.

Observe $\exists$ a nested sequence

$$C \supseteq C_1 \supseteq C_2 \supseteq \cdots \supseteq C_k \supseteq \cdots$$

where $C_k := \{x \in C \mid$ for every $1 \le i \le k$ all ith order partials of every component of F vanish at $x\}$.

(note: each C_k, C closed subset of U).

This is a direct consequence of:

Step 1: $F(C \setminus C_1)$ has measure zero.

Step 2: $F(C_k \setminus C_{k+1})$ has measure zero.

(omitted, similar to step 1).

Step 3: For $k \gg 0$, $F(C_k)$ has measure zero.

Sketch of step 1: $\qquad F = (F_1, \dots, F_n) : \underset{\mathbb{R}^n}{\overset{}{}} U \to V \quad (U \subset \mathbb{R}^n,\ V \subset \mathbb{R}^n)$.

By replacing U, domain of F by $U \setminus C_1$, we just assume that $C_1 = \emptyset$ & show in that case that $F(C)$ has measure 0.

Say $a \in C$. Since $a \notin C_1$ by definition some partial derivative of $F \neq 0$ at a, WLOG (by rearranging coords) can assume $\frac{\partial F_1}{\partial x_1}(a) \neq 0$.

Consider the map $h : \mathbb{R}^n \to \mathbb{R}^n$, $h = (F_1(\vec{x}), x_2, \dots, x_n)$.

Note: $dh(a)$ is non-singular $\quad dh(a) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(a) \neq 0 & 0 \\ * & I_{n-1} \end{pmatrix}$.

By shrinking (IFT) domain, h gives a diffeo: $a \in \tilde{U} \xrightarrow[h]{\cong} \tilde{\tilde{U}} \subset_{\text{open}} \mathbb{R}^n$.

Study $F|_{\tilde{U}}(C) = F \circ h^{-1}(\tilde{C})$. critical set of $F \circ h^{-1}$.

Now, $F \circ h^{-1} = (x_1, F_2(\vec{x}), \dots, F_n(\vec{x}))$.

$$\Rightarrow \quad d(\tilde{F} = F \circ h^{-1}) = \begin{pmatrix} 1 & 0 \\ * & \left(\frac{\partial F_i}{\partial x_j}\right)_{i=2, j=2}^{m,n} \end{pmatrix}.$$

$\Rightarrow \tilde{F}$ has a crit. point at $x_1 = c$, $x_2, \dots, x_m = \bar{x}_{rest}$ iff

$F_{small} = (F_2|_{x_1=c}, \dots, F_n|_{x_1=c})$ has a critical point at $\bar{x}_{rest}$.

also, $\tilde{F}(\tilde{C} \cap \{x_1 = c\}) = F_{small}(\text{critical set of } F_{small}) \cap \{y_1 = c\}$.

$\Rightarrow$ by induction since $\tilde{F}(\tilde{C}) \cap (\{c\} \times \mathbb{R}^{n-1})$ has measure 0, so does $\tilde{F}(\tilde{C})$.

$\Rightarrow$ so does $F|_{\tilde{U}}(C \cap \tilde{U})$.

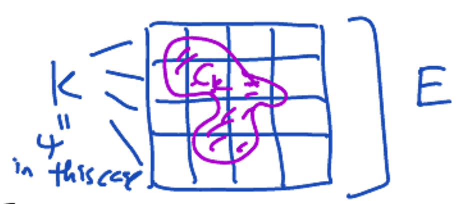
(exercise) $\Rightarrow F(C)$ has measure 0. $\square$.

A few words about <u>Step 3:</u> For $k >> 0$ $(k > m/n - 1)$, $F(C_k)$ has measure 0.

- reduce to case domain U is a cube E.
- use Taylor's theorem to bound the volume of $F(C_k)$

More precisely, on E:

- pick A a constant such that $|\partial^{\vec{v}} f| \leq A$ over E for all $\vec{v}$ of size $k+1$.
- let $R =$ sidelength(E)
- K some large integer



subdivide E into K^m cubes of sidelength R/K, $E_1, \dots, E_{K^m}$.

Over such E_i, if E_i intersects C_k, then for any point $a_i \in C_k \cap E_i$,

Taylor's thm (bounding remainder)

bounded by $2R/K$ on E_i.

$$\Rightarrow \text{ over } E_i, \quad |F(x) - F(a_i)| \leq A' |x - a_i|^{k+1}.$$

const. <u>only</u> depending on A, k, m.

$\Rightarrow$ for any i,

$F(C_k \cap E_i)$ is contained in a cube of sidelength $4A'(R/K)^{k+1}$.

$\Rightarrow F(C_k)$ contained in a union of cubes of total volume

$$\underbrace{K^m}_{\#\text{cubes}} \cdot \left(4A'(R/K)^{k+1}\right)^n.$$

$$= \underbrace{A''}_{(4A'R^{k+1})^n} \cdot K^{m-(k+1)n} = A'' K^{m-nk-n}.$$

If $k > \frac{m}{n} - 1$, then $m - nk - n < 0$, so by taking K arbitrarily large, we can ensure $F(C_k)$ contained in a union of cubes of arbitrarily small volume. $\boxtimes$