• Quickly finish sketch of Sard's thm.

Application of Sard's theorem: embedding manifolds in small $\mathbb{R}^N$'s.

Thm: (Whitney): Let M^m be compact (can be removed.), $\dim(M)=m$. Then, $\exists$ an embedding of $M^m \hookrightarrow \mathbb{R}^{2m+1}$.

(Rmk: 'strong Whitney': says $M^m \hookrightarrow \mathbb{R}^{2m}$, then can also obtain better depending on m and on M)

Pf: Starting point is Thm from earlier which says $M^m \hookrightarrow \mathbb{R}^N$ for $N \gg 0$.

Downward induction to reduce N. Suppose have an embedding $f: M^m \hookrightarrow \mathbb{R}^N$, $N > 2m+1$.

For every $\vec{v} \in S^{N-1} \subset \mathbb{R}^N$, let

$$H_{\vec{v}} = \{\vec{w} \mid \vec{w} \cdot \vec{v} = 0\}$$ be the hyperplane orthogonal to $\vec{v}$, &

$\cong \mathbb{R}^{N-1}$

$$\pi_{\vec{v}} : \mathbb{R}^N \to H_{\vec{v}} \text{ orthogonal projection:}$$

$$\vec{x} \longmapsto \vec{x} - \frac{\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}.$$

The following lemma immediately proves thm by induction. $\boxtimes$

Lemma: If $N > 2m+1$, then the set $\{\vec{v} \in S^{N-1}$ such that $\pi_{\vec{v}} \circ f : M \to H_{\vec{v}} \cong \mathbb{R}^{N-1}$ is still an embedding$\}$ is of full measure in S^{N-1} meaning the complement has measure zero.

(in particular, this set is dense; in particular it's non-empty).

Pf of Lemma:

Given our embedding $f: M \hookrightarrow \mathbb{R}^N$, consider:

diagonal, $\Delta = \{(x,x) \in M \times M \mid x \in M\}$.

$$F: M \times M \setminus \Delta \longrightarrow S^{N-1}$$

$$(x,y) \longmapsto \frac{f(x)-f(y)}{\|f(x)-f(y)\|}$$

$\uparrow x \neq y$

well-defined, i.e., denominator is non-zero, b/c f is injective.

note: $\dim(M \times M \setminus \Delta) = \dim(\mathring{TM}) = 2m$.

$\mathring{TM} = \{(x,w) \in TM \mid w \in M, v \neq 0\}$.

and $$G: (\mathring{TM}) \longrightarrow S^{N-1}$$

$$(x,w) \longmapsto \frac{df_x(w)}{\|df_x(w)\|}$$ $\longleftarrow$ $df_x : T_xM \to T_x\mathbb{R}^N \cong \mathbb{R}^N$ is injective, b/c f is an immersion. Hence, $df_x(w) \neq 0$ if $w \neq 0$, so this is well-defined.

Observe: $\pi_{\vec{v}} \circ f : M \to H_{\vec{v}}$ is

- injective iff $\vec{v} \notin \operatorname{im}(F)$

  (b/c if $\pi_{\vec{v}} \circ f(x) = \pi_v \circ f(y)$ then $f(x) - f(y)$ is a multiple of $\vec{v}$)

- immersion iff $\vec{v} \notin \operatorname{im}(G)$

  (why? exercise).

Now domains of F and G are manifolds of dimension $2m < \underbrace{N-1}_{\dim(S^{N-1})}$ by hypothesis.

Hence Sard's theorem implies that $\operatorname{im}(F)$ & $\operatorname{im}(G)$ must have measure 0 in S^{N-1},

all of $\operatorname{im}(F)$ is critical value, similar with $\operatorname{im}(G)$ b/c of $2m < N-1$.

so $\operatorname{im}(F) \cup \operatorname{im}(G)$ has measure 0 too.

So the complement $\{\vec{v} \in S^{N-1} \mid \vec{v} \notin \operatorname{im}(F), \vec{v} \notin \operatorname{im}(G)\} \overset{\text{above}}{=} \{\vec{v} \in S^{N-1} \mid \pi_{\vec{v}} \circ f \text{ is an embedding}\}$ has full measure.

Rmk: $\pi_v \circ f$ is proper for such $\vec{v}$ automatically (exercise).

$\boxed{\text{E}}$

A slight variation of these arguments (studying restriction of G to $T^1M = \{(x,w) \mid x \in M, \|df_x(w)\| = 1\}$)

$\Rightarrow$ Thm (Whitney immersion theorem): If M^m compact, $\exists$ immersion $g : M \to \mathbb{R}^{2m-1}$.

(exercise.)

Towards vector fields and 1-forms

vector field: "smoothly varying collection of tangent vectors" need TM ✓

1-form: "smoothly varying collection of cotangent vectors" need T^*M. (now).

recall have T_p^*M, $\ni$ cotangent vector. reviewed below

The <u>cotangent bundle</u>: M^n, $p \in M$.

Recall can define T_p^*M as $\mathcal{J}_p / \mathcal{J}_p^2$, where $\mathcal{J}_p \subseteq C^\infty(p)$ ideal of germs of functions which vanish at p.

(we had defined T_pM as $(\mathcal{J}_p/\mathcal{J}_p^2)^*$, one def'n),

we could also define it as $T_p^*M := (T_pM)^*$ ← linear dual, using favorite method to define T_pM.

Given any function $f \in C^\infty(p)$ (really $f = [(f,U)]$) (e.g., the restriction of $\tilde{f}: M \to \mathbb{R}$),

$f - f(p)$ gives an element of $\mathcal{J}_p$, hence induces an element $[f - f(p)] \in T_p^*M$;

↑ constant function.

call this element $df(p)$ or $df_p \in T_p^*M$.

Q: How does this compare to what we've been calling $df_p : T_pM \to T_p\mathbb{R} \cong \mathbb{R}$, $df_p \in (T_pM)^*$?

(Ans – exercise: it's the <u>same</u>.),

This gives a map $d : C^\infty(p) \to T_p^*M$, (exercise: see the map d via all definitions of T_p^*M)

<u>Next</u> time: topology/manifold str. on $T^*M = \{(x, \alpha) \mid x \in M,\ \alpha \in T_x^*M\}$

↗ <u>cotangent bundle</u>.