## The cotangent bundle (cont'd)

$M$, $p \in M$ $\rightsquigarrow$ $T_p^*M$ cotangent vector space (dual to $T_pM$)

- (directly define as e.g., $\mathcal{F}_p/\mathcal{F}_p^2$, $\mathcal{F}_p \subseteq C^\infty(p)$ functions vanishing at $p$)
- Given any function defined near $p$ $f$ inducing $[f] \in C^\infty(p)$, can take $df(p) \in T_p^*M$.

$$\begin{gathered} df(p) \\ \shortparallel \\ [f - f(p)] \in \mathcal{F}_p/\mathcal{F}_p^2. \end{gathered}$$

- gives $d: C^\infty(p) \to T_p^*M$.

<u>check:</u> If $x_1, \dots, x_n$ local coordinates defined at $p$. (i.e., $x_i := x_i \circ \phi$, $(U, \phi)$ chart around $p$).

then $dx_1(p), \dots, dx_n(p)$ are linearly independent and a basis for $T_p^*M$.

(analogously to $TM$,)

Let's now topologize $T^*M = \coprod_{p \in M} T_p^*M = \{(p, v^*),\ p \in M,\ v^* \in T_p^*M\}$.

comes w/ $\pi: T^*M \to M$

$(p, v^*) \longmapsto p$.

as before:

Given $\mathcal{A} = \{(U_i, \phi_i)\}_{i \in I}$ atlas for $M$ in $M$'s differentiable structure,

define $\tilde{U}_i = \pi^{-1}(U_i)$, and

$$\tilde{\phi}_i : \tilde{U}_i = \pi^{-1}(U_i) \longrightarrow \phi_i(U_i) \times \mathbb{R}^m \underset{\text{open}}{\subseteq} \mathbb{R}^m \times \mathbb{R}^m$$

$$(p, v^*) \longmapsto \left(\phi_i(p),\ d(\phi_i^{-1})^*_{\phi_i(p)}(v^*)\right)$$

$v^* \in T_p^*M = (T_pM)^*$.

$\phi_i^{-1} : \phi_i(U_i) \to U_i$

$d(\phi_i^{-1})_{\phi_i(p)} : T_{\phi_i(p)}(\phi_i(U_i)) \to T_pU_i$, where $T_{\phi_i(p)}(\phi_i(U_i)) \cong \mathbb{R}^m$ and $T_pU_i = T_pM$.

and $d(\phi_i^{-1})^*_{\phi_i(p)} : T_p^*M \to T^*_{\phi_i(p)}(\phi_i(U_i))$

$\cong$

$(\mathbb{R}^m)^*$

$\cong$

$\mathbb{R}^m$.

Use the $\{(\tilde{U}_i, \tilde{\phi}_i)\}$ to define

- a topology on $T^*M$

  (determined by declaring that each $\tilde{U}_i$ is open and each $\tilde{\phi}_i$ is a homeo. onto $\phi_i(U_i) \times \mathbb{R}^m$ w/ usual topology)

- a smooth manifold structure on $T^*M$.

<u>Exercise</u> (working out details above): Show $T^*M$ is a smooth manifold, and $\pi: T^*M \to M$ is a smooth map..

(for the latter note that using the charts $(\tilde{U}_i, \tilde{\phi}_i)$ on $T^*M$ and $(U_i, \phi_i)$ on $M$, $\pi$ becomes

$\phi_i \circ \pi \circ \tilde{\phi}_i^{-1} : \phi_i(U_i) \times \mathbb{R}^m \xrightarrow[\text{proj.}]{} \phi_i(U_i)$, which is smooth).

<u>Functoriality</u>: Let $f: M \to N$ be a smooth map.

we've seen that this induces, at any $p \in M$, an alg. hom.

$$f^* := (-)\circ f : C^\infty(f(p)) \to C^\infty(p)$$

$$[g] \longmapsto [g \circ f]$$

$\mathcal{F}_{f(p)} \longrightarrow \mathcal{F}_p$

$\mathcal{F}^2_{f(p)} \longrightarrow \mathcal{F}^2_p$

and hence maps

- $f_* = (f_*)_p = df_p : T_pM \longrightarrow T_{f(p)}N$

  $X \longmapsto X \circ f^*$.

  new notation for

- $f^* = (f^*)_p : T^*_{f(p)}N \longrightarrow T^*_pM$.

  (direct from $f^*$ ~~def~~ above by noticing it sends $\mathcal{F}_{f(p)}$ to $\mathcal{F}_p$ & $\mathcal{F}^2_{f(p)}$ to $\mathcal{F}^2_p$.)

  OR dual of $(f_*)_p$.

More generally (<u>exercise</u>) check $f$ induces a $C^\infty$ map

$$f_* \text{ or } df : TM \longrightarrow TN \quad (\underline{\text{exercise}}).$$

$$(p,v) \longmapsto (f(p), df_p(v))$$

compatible w/ projection, meaning

$$\begin{tikzcd} TM \arrow[r, "f_*"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & TN \arrow[d, "\pi"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

but not always a map $T^*N \dashrightarrow T^*M$ 

if $q = f(p)$ for a unique $p$.

$(q, w^*) \overset{?}{\longmapsto} (f^{-1}(q)??? , f^*_{f(p)}(w^*))$

$w^* \in T^*_q N$

$f$ may not have a two-sided inverse

(works if $f$ is a diffeomorphism, and if $f$ is a diffeo., both $f_*$ and $f^*$ are diffeomorphisms also.)

## 1-forms and vector fields

**Def:** Let $T^*M \xrightarrow{\pi} M$ be the cotangent bundle. A 1-form over an open $U \subseteq M$ is a smooth map $U \xrightarrow{s} T^*M$ such that $\underbrace{\pi \circ s = id_U}_{(\star)} : U \hookrightarrow M$.

$(\star)$ implies that $s$ must have the form

$$p \longmapsto (p, s_p), \quad s_p \in T^*_p M.$$

i.e., $s$ assigns to each $p \in U$ an element of $T^*_p M$ in a "smoothly varying" fashion.

The space of 1-forms on $U$ is denoted $\Omega^1(U)$

1-forms on $M$ (case $U = M$) : $\Omega^1(M)$.

**Note:** $\Omega^1(U)$ is a vector space$/\mathbb{R}$, & moreover a module over algebra $C^\infty(U)$.

why: $(s+s')_p := s_p + s'_p$ using $+$ on $T^*_p M$.

if $f \in C^\infty(U)$ (e.g., constant function $c \in \mathbb{R}$), $s \in \Omega^1(U)$,

then $fs : p \longmapsto (p, f(p) \cdot s_p)$

uses scalar mult in $T^*_p M$.

$(fs)_p = f(p) s_p$.

1) we often write $\Omega^0(U) = C^\infty(U)$ "zero forms". ($U = M$ allowed)

There is a map $d : \Omega^0(U) \to \Omega^1(U)$

$$g \longmapsto dg$$

1-form given by $p \longmapsto (p, dg(p))$, $dg(p) \in T_p^*M$

(exercise: if $g \in C^\infty(U)$, then $dg$ as defined is a smooth map $U \to T^*M$, i.e., $dg \in \Omega^1(U)$).

(later: $\Omega^p(M)$ or $U$ $\forall\, p \leq \dim M = m$ : "$p$-forms")

2) Given a smooth $f: M \to N$, we've seen there may not exist $f^*: T^*N \to T^*M$; nevertheless $\exists$ $f^*: \Omega^1(N) \longrightarrow \Omega^1(M)$

$$\alpha \longmapsto f^*\alpha$$

"pullback of $\alpha$ along $f$":

$$f^*(\alpha): p \longmapsto (p, f_p^*(\alpha_{f(p)})),$$

$\alpha_{f(p)} \in T^*_{f(p)}N$

$f_p^*: T^*_{f(p)}N \to T_p^*M$

3) Given $f: M \to N$, $g: N \to P$, $\alpha \in \Omega^1(P)$,

<u>exercise</u>: $$(g \circ f)^*\alpha = f^*g^*\alpha \quad \in \Omega^1(M).$$

(i.e., $\mathrm{Man}^{op} \longrightarrow \mathrm{Vect}_{\mathbb{R}}$ is a <u>contravariant</u> functor.

$M \longmapsto \Omega^1(M)$

$\{f: M \to N\} \longmapsto f^*$.)

4) $f$ as before: $M \xrightarrow{f} N$, then this diagram commutes:

$$\begin{array}{ccc} \Omega^0(N) & \xrightarrow{f^*} & \Omega^0(M) \\ \downarrow d & \circlearrowright & \downarrow d \\ \Omega^1(N) & \xrightarrow{f^*} & \Omega^1(M) \end{array}$$

5) $d(gh) = g\,dh + h\,dg$.

Ex: 1-form on $\mathbb{R}^2$

$$\alpha = x^2\,dy + y\,dx \in \Omega^1(\mathbb{R}^2)$$

means: $\left(p=(x,y) \longmapsto \left(p=(x,y),\ v^* = x^2\,dy|_p + y^2\,dx|_p\right)\right)$

Consider $i : \mathbb{R} \longrightarrow \mathbb{R}^2$

$$t \longmapsto (t,0)$$

Then
$$i^*\alpha \overset{?}{=} \overbrace{i^*x^2}^{t^2}\ \underbrace{i^*dy}_{= d(i^*y) = d(0) = 0} + \overbrace{i^*y}^{0}\ \underbrace{i^*dx}_{= d(i^*x) = d(t) = dt.} = 0.$$

Next time: A vector field is similarly a smooth $M \xrightarrow{X} TM$ w/ $\pi \circ X = \mathrm{id}_M$.