

The cotangent bundle (cont'd)

$M, p \in M \rightsquigarrow T_p^*M$ cotangent vector space (dual to T_pM)

- (directly define as e.g., $\mathcal{F}_p/\mathcal{F}_p^2$, $\mathcal{F}_p \subseteq C^\infty(p)$ functions vanishing at p)

- Given any function defined near p f inducing $[f] \in C^\infty(p)$, can take $df(p) \in T_p^*M$.

$$[f - f(p)] \in \mathcal{F}_p/\mathcal{F}_p^2.$$

- gives $d: C^\infty(p) \rightarrow T_p^*M$.

check: If x_1, \dots, x_n local coordinates defined at p . (i.e., $x_i = x_i \circ \phi$, (ϕ) chart around p).

then $dx_1(p), \dots, dx_n(p)$ are linearly independent and a basis for T_p^*M .

(analogously to TM),

Let's now topologize $T^*M = \coprod_{p \in M} T_p^*M = \{(p, v^*), p \in M, v^* \in T_p^*M\}$.

comes w/ $\pi: T^*M \rightarrow M$
 $(p, v^*) \mapsto p$.

as before:

Given $\mathcal{A} = \{(U_i, \phi_i)\}_{i \in I}$ atlas for M in M 's differentiable structure,

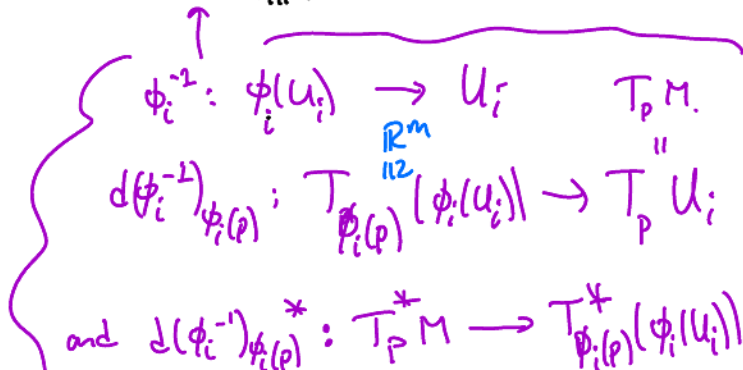
define $\tilde{U}_i = \pi^{-1}(U_i)$, and

$\mathbb{R}^m \times \mathbb{R}^m$
 U_i open

$$\tilde{\phi}_i: \tilde{U}_i = \pi^{-1}(U_i) \rightarrow \phi_i(U_i) \times \mathbb{R}^m$$

$$(p, v^*) \mapsto (\phi_i(p), d(\phi_i^{-1})_{\phi_i(p)}^*(v^*))$$

\uparrow
 T_p^*M
 $(T_p^*M)^*$.

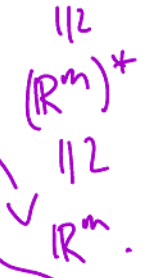


Use the $\{(\tilde{U}_i, \tilde{\phi}_i)\}$ to define

- a topology on T^*M

(determined by declaring that each \tilde{U}_i is open and each $\tilde{\phi}_i$ is a homeo. onto $\phi_i(U_i) \times \mathbb{R}^n$ w/ usual topology)

- a smooth manifold structure on T^*M .



Exercise (working out details above): Show T^*M is a smooth manifold, and $\pi: T^*M \rightarrow M$ is a smooth map.

(for the latter note that using the charts $(\tilde{U}_i, \tilde{\phi}_i)$ on T^*M and (U_i, ϕ_i) on M , π becomes

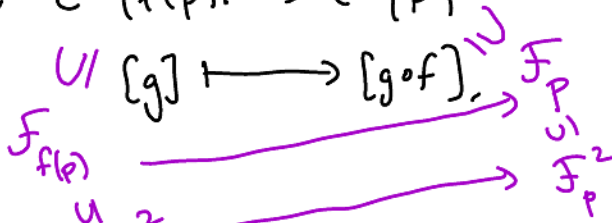
$$\phi_i \circ \pi \circ \tilde{\phi}_i^{-1} : \phi_i(U_i) \times \mathbb{R}^n \xrightarrow{\text{proj.}} \phi_i(U_i), \text{ which is smooth.}$$

Functoriality: Let $f: M \rightarrow N$ be a smooth map.

we've seen that this induces, at any $p \in M$, an alg. hom.

$$f^* := (-) \circ f : C^\infty(f(p)) \rightarrow C^\infty(p)$$

and hence maps



$$f_* = (f_*)_p = df_p : T_p M \rightarrow T_{f(p)} N$$

$$X \mapsto X \circ f^*$$

$$f^* = (f^*)_p : T_{f(p)}^* N \rightarrow T_p^* M$$

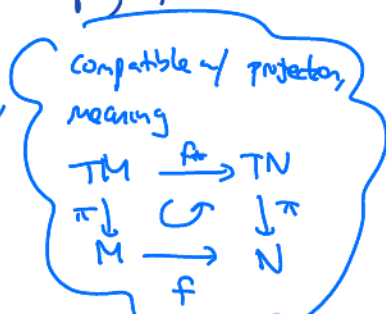
(direct from f^* above by noting it sends $F_{f(p)}^1$ to F_p^1 & $F_{f(p)}^2$ to F_p^2)

OR dual of $(f_*)_p$

More generally (exercise) check f induces a C^∞ map

$$f_* \text{ or } df : TM \rightarrow TN \quad (\text{exercise}).$$

$(p, v) \mapsto (f(p), df_p(v))$



but not always a map $T^*N \rightarrow T^*M$
 $(q, w^*) \mapsto (f^{-1}(q), \dots)$ if $q = f(p)$ for a unique p .
 $f^*_{f(p)}(w^*)$
 f may not have a two-sided inverse

(works if f is a diffeomorphism, and ~~is~~ if f is a diffeo., both f_* and f^* are diffeomorphisms also.)

1-forms and vector fields

Def: Let $T^*M \xrightarrow{\pi} M$ be the cotangent bundle. A 1-form over an open $U \subseteq M$ is a smooth map $U \rightarrow T^*M$ such that $\pi \circ s = \text{id}_U$: $U \hookrightarrow M$.

(*) implies that s must have the form

$$p \mapsto (p, s_p) \\ \uparrow \\ T_p^*M.$$

i.e., s assigns to each $p \in U$ an element of T_p^*M in a "smoothly varying" fashion.

The space of 1-forms on U is denoted $\Omega^1(U)$

1-forms on M (case $U=M$): $\Omega^1(M)$.

Note: $\Omega^1(U)$ is a vector space / \mathbb{R} , & moreover a module over algebra $C^\infty(U)$.

why: $(s+s')_p := s_p + s'_p$ using $+$ on T_p^*M .

if $f \in C^\infty(U)$ (e.g., constant function $c \in \mathbb{R}$), $s \in \Omega^1(U)$,

then $fs : p \mapsto (p, f(p) \cdot s_p)$

$$(fs)_p = f(p) s_p.$$

\uparrow uses scalar mult in T_p^*M .

1) we often write $\Omega^0(U) = C^\infty(U)$ "zero forms";

($U=M$ allowed)

There is a map $d: \Omega^0(U) \rightarrow \Omega^1(U)$

$$g \longmapsto dg$$

(exercise: if $g \in C^\infty(U)$, then dg as defined is a smooth map $U \rightarrow T^*M$,
i.e., $dg \in \Omega^1(U)$).

T_p^*M

1-form given by $p \mapsto (p, dg(p))$

(later: $\Omega^p(M) \forall p \leq \dim M = m$: "p-forms")
or U

2) Given a smooth $f: M \rightarrow N$, we've seen there may not exist $f^*: T^*N \rightarrow T^*M$;

nevertheless $\exists f^*: \Omega^1(N) \rightarrow \Omega^1(M)$

$$\alpha \longmapsto f^* \alpha$$

~ pullback of α along f :

$$f^*(\alpha) = p \mapsto (p, f_p^*(\alpha_{f(p)}))$$

$T_{f(p)}^*N$

$$f_p^*: T_{f(p)}^*N \rightarrow T_p^*M$$

3) Given $f: M \rightarrow N$, $g: N \rightarrow P$, $\alpha \in \Omega^1(P)$,

exercise: $(g \circ f)^* \alpha = f^* g^* \alpha \in \Omega^1(M)$.

(i.e., $\text{Map}^p \rightarrow \text{Vect}_p$ is a contravariant functor.

$$M \longmapsto \Omega^1(M)$$

$$\{f: M \rightarrow N\} \longmapsto f^* .)$$

4) f as before: $M \xrightarrow{f} N$, then this diagram commutes:

$$\begin{array}{ccc} \Omega^0(N) & \xrightarrow{f^*} & \Omega^0(M) \\ \downarrow \text{id} & \curvearrowright & \downarrow \text{id} \\ \Omega^1(N) & \xrightarrow{f^*} & \Omega^1(M) \end{array}$$

$$5) d(gh) = g dh + h dg.$$

Ex: 1-form on \mathbb{R}^2

$$\alpha = x^2 dy + y dx \in \Omega^1(\mathbb{R}^2)$$

↑ means: $(p=(x,y)) \mapsto (p=(x,y), v^* = x^2 dy|_p + y^2 dx|_p)$

Consider $i: \mathbb{R} \rightarrow \mathbb{R}^2$

$$t \mapsto (t, 0) \quad t^2$$

$$\text{Then } i^* \alpha \stackrel{?}{=} \overbrace{i^* x^2}^{t^2} \underbrace{i^* dy}_{\parallel d(i^* y)} + \overbrace{i^* y}^0 \underbrace{i^* dx}_{\parallel d(i^* x)} = 0.$$

$$\begin{array}{ccc} \parallel & & \parallel \\ d(0) & & d(t) \\ \parallel & & \parallel \\ 0 & & dt. \end{array}$$

Next time: A vector field is similarly a smooth $M \xrightarrow{X} TM$ w/ $\pi \circ X = id_M$.