Vector fields

A <u>vector field</u> on M is a smooth map $X: M \longrightarrow TM$ such that $\pi \circ X = id_M$ $(\pi: TM \to M)$ $\rightsquigarrow$ X is of the form $p \longmapsto (p, X_p)$

(<u>Rmk</u>: both TM & T^*M are special cases of notion of a <u>vector bundle</u> $\begin{matrix} E \\ \downarrow \pi \\ M \end{matrix}$, & vector field / one-forms are <u>sections</u> of TM / T^*M respectively; general notion of sections of any vector bundle: a smooth map $s: M \to E$ w/ $\pi \circ s = id_M$).

<u>Note</u>:
- can also have a vec. field over an open $U \subseteq M$.
- The space of vector fields on M, denoted $\mathfrak{X}(M)$, is a vector space/$\mathbb{R}$ & moreover a $C^\infty(M)$-module (sections of any vector bundle are a $C^\infty(M)$-module.)

<u>Example</u>:

On $\mathbb{R}^n$,

$$\frac{\partial}{\partial x_i}: \mathbb{R}^n \longrightarrow T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \quad \text{is a vector field.}$$

$$(p, \textstyle\sum a_i \frac{\partial}{\partial x_i}(-)\big|_{x=p}) \longleftrightarrow (p, \vec{a})$$

$$p \longmapsto (p, \underbrace{\frac{\partial}{\partial x_i}(-)\big|_{x=p}}_{\in T_p\mathbb{R}^n}) \cong (p, \vec{e}_i)$$

More generally, any vector field on an open $U \subseteq \mathbb{R}^n$ must be of the form

$$X: U \longrightarrow U \times \mathbb{R}^n \cong TU.$$

$$p \longmapsto (p, f(p) = (f_1(p), \ldots, f_n(p))) \cong \sum f_i(p) \frac{\partial}{\partial x_i}.$$

by def'n this is the vector field $p \longmapsto (p, \sum f_i(p) \frac{\partial}{\partial x_i}(-)\big|_{x=p})$.

or in shorthand,

$$\underline{X = \sum f_i \frac{\partial}{\partial x_i}}, \quad f_i \in C^\infty(U)$$

(analogously, any 1-form on $U \underset{\text{open}}{\subseteq} \mathbb{R}^n$ is of the form

$$\sum g_i dx_i, \text{ where } g_i \in C^\infty(U)).$$

In particular, we obtain from above a way to express any vector field (resp. 1-form) on M in local coords near $p \in M$.

Often draw vector fields:

e.g., on $\mathbb{R}^2$

Integrating vector fields

Def: An integral curve of a vector field $X \in \mathfrak{X}(M)$ is a parametrized curve

$$\underset{\substack{\cap \text{ open} \\ \mathbb{R}}}{\gamma : I} \longrightarrow M \quad \text{satisfying} \quad \dot{\gamma}(a) = \underset{\substack{\Uparrow \\ T_{\gamma(a)}M}}{X_{\gamma(a)}} \quad \text{for all } a \in I.$$

can also think of as $d\gamma_a\left(\frac{\partial}{\partial t}\right)$, or $[\gamma(t+a)] \in T_{\gamma(a)}M = C_{\gamma(a)}/\sim$.

$X : M \to TM$
$p \mapsto (p, \underset{\substack{\Uparrow \\ T_pM}}{X_p})$.

Q: Given X, does an integral curve for X $\gamma : (-\varepsilon, \varepsilon) \to M$ exist with $\gamma(0) = p$ ("initial conditions") for any fixed p? If so does γ "vary smoothly" as we vary p in some sense?

Thm: [variant of "Fundamental thm of flows / ODEs" from Lee]:

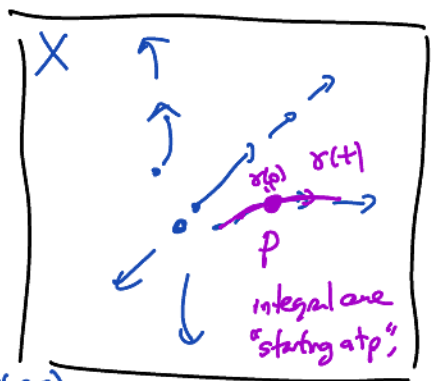
X any vector field, $p \in M$ any point, then $\exists$ open $U \ni p$ and $\varepsilon > 0$, along with a smooth map "local flow" ("space of initial conditions")

$$\Phi : U \times (-\varepsilon, \varepsilon) \to M$$

with $\Phi(q, 0) = q$ for all $q \in U$,

and $\frac{d}{dt}\Phi(q, -)\Big|_{t=a} = X_{\Phi(q,a)}$.

Moreover, this is unique in the sense that any other local flow agrees with this one on common domain of definition.



Rmk: if $s, t, s+t \in (-\varepsilon, \varepsilon)$, then $\Phi(\Phi(q,s), t) = \Phi(q, s+t)$. Follows from uniqueness (exercise).

(write $\gamma_q := \Phi(q, -)$, this is $\gamma_q(0) = q$, & $\dot{\gamma}_q(a) = X_{\gamma_q(a)}$.

Saying Φ is smooth expresses smooth dependence of γ_q on q).

This reduces quite directly to ordinary ODE theory ("Fundamental theorem of ODEs") as follows [we'll omit the proof, but see [Lee-Appendix]).

Pf reduces to the case $M = W \underset{\text{open}}{\subset} \mathbb{R}^m$ by using a chart.

On W, our vector field $X = \sum f_i \frac{\partial}{\partial x_i}$, and at $p \in W$, we're seeking a smaller $\underset{p}{\overset{\widetilde{W}}{\cup}}$ & $^{\varepsilon>0}$ a map $\Phi : \widetilde{W} \times (-\varepsilon, \varepsilon) \to W$ satisfying

$$\Phi(q,0) = \gamma_q(0) = q.$$

$$\dot{\gamma}_q(a) = X_{\gamma_q(a)} = \sum f_i(\gamma_q(a)) \frac{\partial}{\partial x_i}$$

$$\begin{bmatrix} \dot{\gamma}_q^1(a) \\ \vdots \\ \dot{\gamma}_q^m(a) \end{bmatrix} = \begin{bmatrix} f_1(\gamma_q(a)) \\ \vdots \\ f_m(\gamma_q(a)) \end{bmatrix}$$

This is a system of ODEs for each q. $\boxtimes$

Global flows:

Def: A (global) flow on M is a C^∞ map

"a smooth action of $\mathbb{R}$ on M"

$$\Phi : M \times \mathbb{R} \longrightarrow M \text{ satisfying}$$

$(*)$ $\Phi(\Phi(p,t), t') = \Phi(p, t+t')$

maybe redundant?

$(**)$ $\Phi(m, 0) = m$.

This is better expressed in terms of $\varphi_t : M \to M$ given by $\varphi_t(x) = \Phi(x,t)$.

Then $(*) \iff \varphi_{t+t'} = \varphi_t \circ \varphi_{t'}$.

$(**) \iff \varphi_0 = \mathrm{id}_M$

$\Rightarrow$ each φ_t is a diffeo. w/ inverse φ_{-t}.

A flow determines a vector field X via the property that

$$X_p = \frac{d}{dt}\varphi_t(p)\Big|_{t=0}. \quad \left(= \frac{d}{dt}\Phi(p,-)\Big|_{t=0}\right).$$

Correspondingly, frequently but not always a vector field determines a flow:

In local flow theorem, if we can take $U = M$ and $\varepsilon = \infty$ then we'd get a map $\Phi : M \times \mathbb{R} \longrightarrow M$

w/ $\Phi(q,0) = q$. & $\Phi(\Phi(q,s),t) = \Phi(q, s+t)$ $\forall\, s,t$.

<u>Non-ex</u>: On $M = \mathbb{R}\setminus 0$, consider $X = \frac{\partial}{\partial x}$.

On $\mathbb{R}$ X integrates to the flow $\Phi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $(x,t) \longmapsto x+t$. $\quad \varphi_t(x) = x+t$.

But when remove 0, $\nexists$ a globally defined $\Phi : (\mathbb{R}\setminus 0)\times(-\varepsilon,\varepsilon) \longrightarrow \mathbb{R}\setminus 0$ for any ε.