From last time:

Takehome midterm: assigned after spring break, deadline ~ 7 days.

Def: A flow on M is a C^∞ map

$$\Phi : M \times \mathbb{R} \longrightarrow M \quad \text{satisfying}$$

$$(*) \qquad \Phi(\Phi(q,t), t') = \Phi(p, t+t')$$

$$(**) \qquad \Phi(m, 0) = m.$$

Denoting $\varphi_t : M \to M$ by $\varphi_t := \Phi(-, t)$:

Then $(*) \iff \varphi_{t+t'} = \varphi_t \circ \varphi_{t'}$.

$(**) \iff \varphi_0 = \mathrm{id}_M$

$\Rightarrow$ each φ_t is a diffeo. w/ inverse φ_{-t}.

$T_pM \times T_0\mathbb{R}$
$\shortparallel$
$T_{(p,0)}(M \times \mathbb{R})$

A flow determines a vector field X via the property that

$$X_p = \frac{d}{dt} \varphi_t(p) \Big|_{t=0}. \qquad \left(= \frac{d}{dt} \Phi(p, -) \Big|_{t=0} = d\Phi_{p,0}\left(\overbrace{0, \tfrac{\partial}{\partial t}}^{\sim}\right) \right)$$

Exercise: $\left(\overset{\text{also}}{\frac{d}{dt}} \varphi_t(p)\Big|_{t=a} = \frac{d}{dt}\left(\varphi_{t-a} \circ \varphi_a(p)\right)\Big|_{t=a} \overset{t'=t-a}{=} \left(\frac{d}{dt'} \varphi_{t'}\right)(\varphi_a(p))\Big|_{t'=0} = X_{\varphi_a(p)}. \right)$.

Correspondingly, frequently but not always a vector field determines a flow:

In local flow theorem, if we could take $U = M$ and $\varepsilon = \infty$ then we'd get a map $\Phi : M \times \mathbb{R} \longrightarrow M$

w/ $\Phi(q, 0) = q$. & $\Phi(\Phi(q,s), t) = \Phi(q, s+t)$ $\forall\, s, t$.

On a general M, a given X may not induce a global flow.

(e.g., $M = \mathbb{R}\setminus 0$, $X = \frac{\partial}{\partial t}$ doesn't make a global flow, & on $M = \mathbb{R}$, $\frac{\partial}{\partial t}$ does).

However:

<u>Thm</u>: If M is <u>compact</u>, X any vector field, then there exists a <u>unique flow</u> $(\varphi_t)_{t\in\mathbb{R}}$ such that $\frac{d}{dt}\varphi_t(p)\big|_{t=a} = X_{\varphi_a(p)}$.

Say the flow $(\varphi_t)_{t\in\mathbb{R}}$ "integrates" X.

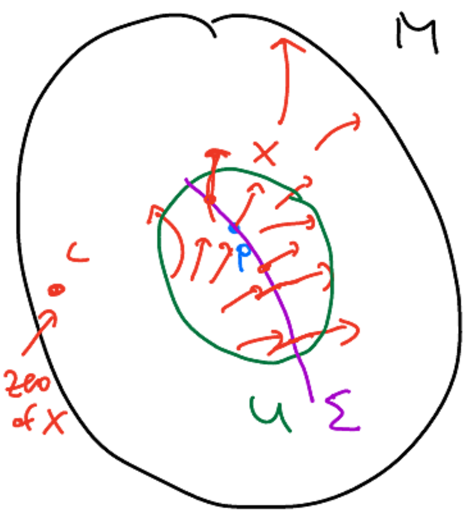
<u>Some corollaries of local & global flows</u>:

$\mathfrak{X}(M) :=$ the space of vector fields on M.

<u>Cor of local flows</u>:

<u>Cor</u>: Say M manifold, $X \in \mathfrak{X}(M)$, and $p \in M$ s.t. $X_p \neq 0$.
Then, there exists a set of coordinates near p such that $X = \frac{\partial}{\partial x_1}$.

<u>Pf</u>: Say $m = \dim(M)$, & pick
- a neighborhood $U \ni p$
- a submanifold $\Sigma^{m-1} \subset U$, $p \in \Sigma$



which is <u>transverse to</u> X, meaning for every $q \in \Sigma$,

$$\underbrace{T_q\Sigma + X_q}_{\text{(i.e., these two span)}} = T_q M$$

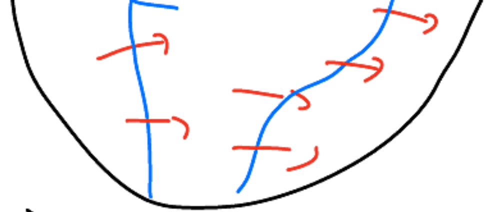
along Σ, never vanishes, & is <u>never tangent</u> to Σ.

(claim: U, Σ satisfying above properties <u>always</u> exist. (why? exercise)).

Now, by shrinking U, Σ if necessary, local flow thm says $\exists\, \varepsilon$ & a flow map $\Phi: U \times (-\varepsilon, \varepsilon) \to M$ integrating X. Restrict to Σ in U to get

$$\overline{\Phi} = \Phi\big|_{\Sigma\times(-\varepsilon,\varepsilon)} : \Sigma \times (-\varepsilon, \varepsilon) \to M.$$

Since Σ is transverse to X, $\overline{\Phi}$ is a diffeomorphism



from some nhood of p onto its image, by inverse function theorem.

(Calculate: $d\overline{\Phi} : T_{(p,0)}\left(\Sigma \times (-\varepsilon,\varepsilon)\right) \xrightarrow{\cong} T_pM$)

$$\overset{\shortparallel}{\mathrm{id}_{T_p\Sigma} \oplus \left(\tfrac{\partial}{\partial t} \mapsto X_p\right)}$$

$\Rightarrow$ can shrink ε to ε', Σ to Σ', U to some $U' = \overline{\Phi}(\Sigma' \times (-\varepsilon', \varepsilon'))$.

s.t. $\overline{\Phi} : \Sigma' \times (-\varepsilon', \varepsilon')_t \xrightarrow[\text{diffeo.}]{\cong} U' \ni p$.

Note that in this coord system (meaning w.r.t. coordinates $t, x_2, \ldots, x_m$ induced by t & $x_2, \ldots, x_m$ coords on Σ),

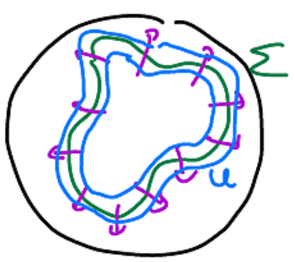
note $X = \dfrac{\partial}{\partial t}$. ☒.

A similar method proves a more global statement if Σ is compact. (pf analogous to global flow existence thm).

Thm ("collar thm"): Given a compact submanifold $\Sigma^{m-1} \subseteq M^m$, and a vector field X which is transverse to Σ everywhere, Flowing by X induces a diffeo,

$$\underbrace{U = \text{nhood}(\Sigma)}_{\text{open in } M} \xleftarrow{\cong} \Sigma \times (-\varepsilon, \varepsilon) \quad \text{for some } \varepsilon, \text{ \& some open nhood } U \text{ of } \Sigma.$$

flow Φ of X restricted to $\Sigma \times (-\varepsilon, \varepsilon)$.



$U \xleftarrow{\cong} \Sigma \times (-\varepsilon, \varepsilon)$.

M

Pf of global flow existence theorem:

Note that if exists a single ε which works for all p in the local flow existence theorem, this implies $\exists$ "uniform short-time flow" $\Phi : M \times (-\varepsilon, \varepsilon) \to M$.

$\Rightarrow$ global flow by iterating short-time flows.

(e.g., $\varphi_T := \varphi_{\frac{2T}{\varepsilon} \cdot \frac{\varepsilon}{2}} = \cdot - $)

Local flow existence says for every p, $\exists$ $U_p \ni p$ & an ε_p that works for that U_p. By compactness $\exists$ a finite subcover $U_{p_1}, \dots, U_{p_k}$ of $\{U_p\}$

$\Rightarrow$ hence get a uniform short-time flow by $\varepsilon = \min(\varepsilon_{p_1}, \dots, \varepsilon_{p_k}) > 0$. $\boxtimes$

Operations involving vector fields

$\mathfrak{X}(M) :=$ vector fields on M. We've previously seen this is an $\mathbb{R}$-vector space & even a $C^\infty(M)$-module.

Vector fields act on functions

There's a map $\mathfrak{X}(M) \times C^\infty(M) \longrightarrow C^\infty(M)$

$$X, \ f \longmapsto X(f)$$

defined by $X(f)(p) := X_p\left(f|_{C^\infty(p)}\right) \in \mathbb{R}$.

$C^\infty(p)$

"directional derivative."

Now, given a pair of vector fields $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, we can take $X(Y(f))$, meaning at p take directional derivative $X_p(Y(f))$.

In general the map

$f|_{C^\infty(p)} \longmapsto X_p\left((Y(f))|_{C^\infty(p)}\right)$ is not a derivation $: C^\infty(p) \to \mathbb{R}$,

because it doesn't satisfy Leibniz. However:

Lemma: $X, Y \in \mathfrak{X}(M)$, then there exist another vector field $[X,Y] \in \mathfrak{X}(M)$, defined by $[X,Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$

for every $f \in C^\infty(M)$ and $p \in M$.

Call $[X,Y]$ the Lie <u>bracket</u> of X and Y.