Last time: M^n manifold ..

Lemma: $X, Y \in \mathfrak{X}(M)$, then there exist another vector field $[X,Y] \in \mathfrak{X}(M)$, defined by $$[X,Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$$

for every $f \in C^\infty(M)$ and $p \in M$.

Call $[X,Y]$ the Lie bracket of X and Y.

Exercise: prove lemma. To check: (a) $[X,Y]_p \in \mathrm{Der}(C^\infty(p), \mathbb{R})$.

(the above defines it as in $\mathrm{Hom}_{\mathbb{R}}(C^\infty(p), \mathbb{R})$)

(b) the map $[X,Y] : M \to TM$
$p \longmapsto (p, [X,Y]_p)$ is smooth, i.e., $[X,Y] \in \mathfrak{X}(M)$.

Prop: The Lie bracket satisfies the following properties:

(1) $[X,Y]$ is linear in both X & Y.

(2) $[Y,X] = -[X,Y]$

(3) $[X, fY] = \underbrace{X(f)}_{\in C^\infty(M)} Y + f[X,Y]$

any $f \in C^\infty(M)$

(from (2)+(3) can also deduce $[fX, Y] = \ldots$)

(4) (Jacobi identity):

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

$\forall\, X, Y, Z \in \mathfrak{X}(M)$.

(Rmk: if had a different association $X, Y \rightsquigarrow \alpha(X,Y)$ vector field
w/ $\alpha(fX, Y) = \alpha(X, fY) = f\alpha(X,Y)$, we say

α is tensorial (not true for $[-,-]$) $\rightsquigarrow$ α comes from a section of the "tensor bundle" $T^*M \otimes T^*M$).

Distributions:

Idea: a nowhere vanishing vector field X determines a "smoothly varying" collection of 1-d subspaces $\mathrm{span}(X_p) \subseteq T_pM$.

Integral curves of X give in particular maps $\gamma: I \to M$ w/

$$d\gamma(T_t I) \subseteq (\text{or} = \text{if immersion}) \ \mathrm{span}(X_{\gamma(t)}) \subseteq T_{\gamma(t)} M.$$

Distributions give a higher dimensional subspace generalization of the above.

Def: M^m. Fix $1 \le c \le m$. A c-dimensional distribution $\mathcal{D}$ on M^m is a choice of c-dimensional subspace $\mathcal{D}_p \subseteq T_pM$ for each $p \in M$.

- Say $\mathcal{D}$ is smooth if, at every $p \in M$, $\exists$ nhood $U \ni p$ and c (smooth) vector fields $X_1, \dots, X_c$ on U which span $\mathcal{D}$ over U.

  (i.e., $\mathrm{span}((X_1)_q, \dots, (X_c)_q) = \mathcal{D}_q$ for every $q \in U$).

  i.e., $\mathcal{D} = \{\mathcal{D}_p\}_{p \in M}$

- Say a vector field X is contained in $\mathcal{D}$, written $X \in \mathcal{D}$ if $X_p \in \mathcal{D}_p$ for every p.

Def: Fix $\mathcal{D}$ a smooth c-dim'l distribution on M^m. An integral submanifold $N^n \subseteq M^m$ of $\mathcal{D}$ is a submanifold where $T_pN \subset \mathcal{D}_p \subseteq T_pM$ for every $p \in N$.

($\Rightarrow \dim N = n \le \dim \mathcal{D} = c$)

Def: $\mathcal{D}$ is integrable if at every p, $\exists$ a chart $(U \ni p, \phi)$ and corresponding local coordinates $x_1, \dots, x_m$ ($x_i := x_i \circ \phi$) such that

$$\mathcal{D} = \mathrm{Span}\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_c}\right)$$

(means $\mathcal{D}_q = \mathrm{Span}\left(\left(\frac{\partial}{\partial x_1}\right)_q, \dots, \left(\frac{\partial}{\partial x_c}\right)_q\right)$ for every q in U)

$\Longleftrightarrow$ near any p $\exists$ functions $f_1, \dots, f_{m-c}$ smooth functions on a nbhd U of p such that for any constants $k_1, \dots, k_{m-c}$, the loci

$$\{ f_1 = k_1, \dots, f_{m-c} = k_{m-c} \}$$ are integral submanifolds of $\mathcal{D}$ with $(f_1, \dots, f_{m-c}) : U \to \mathbb{R}^{m-c}$ is a submersion.

We can specify distributions by:

- specifying c globally independent (i.e., at every point) $X_1, \dots, X_c \in \mathfrak{X}(M)$

  $\rightsquigarrow \mathcal{D} = \mathrm{span}(X_1, \dots, X_c)$

- dually, by specifying a globally independent $\theta_1, \dots, \theta_{m-c} \in \Omega^1(M)$

  $$\rightsquigarrow \mathcal{D} = \bigcap_{i=1}^{m-c} \ker(\theta_i)$$

  (note: each $(\theta_i)_p : T_pM \to \mathbb{R}$)

Ex: $\mathbb{R}^3$, $\omega = dz \in \Omega^1(\mathbb{R}^3)$.

Then $\mathcal{D} = \ker \omega = \mathrm{Span}\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \}$.

The integral surfaces of $\mathcal{D}$ are of the form $\{z = \text{const}\}$.

$\mathcal{D}$ is an integrable 2-plane field distribution.

Ex: $\mathbb{R}^3$, $\omega = dz + (x dy - y dx)$. Then

$$\mathcal{D} = \ker(\omega) \underset{\text{check}}{=} \mathrm{span}\left\{ x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y},\ y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right\}$$

is not integrable. (first example of what's called a contact distribution.)

(exercise).

Q: When is $\mathcal{D}$ integrable?

Case $\dim(\mathcal{D}) = 1$. By ODE theory we know that if $X_p \neq 0$ then $\text{Span}(X)$ is (at least locally near p) integrable. (Last class we showed if $X_p \neq 0$ then $X = \frac{\partial}{\partial x_1}$ in some coords near p)

$\Rightarrow$ any 1-dim'l $\mathcal{D}$ is integrable.

<u>Def</u>: Say $\mathcal{D} \subset TM$ (smooth c-dim'l dist.) is <u>involutive</u> if, whenever $X, Y \in \mathcal{D}$ then $[X, Y] \in \mathcal{D}$.

("$\mathcal{D}$ is closed under $[-,-]$").

<u>Thm</u>: [<u>Frobenius theorem</u>]: $\mathcal{D}$ is integrable <u>if and only if</u> $\mathcal{D}$ is involutive.

<u>Start of proof sketch</u>:

- $\Rightarrow$ Say $\mathcal{D}$ is integrable. Then at each p $\exists$ local coordinates near p s.t. $\mathcal{D} = \text{Span}\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_c}\right)$.

  $\Rightarrow$ Given $X, Y \in \mathcal{D}$ we can write $X = \sum_{i=1}^{c} a_i(x) \frac{\partial}{\partial x_i}$ & $Y = \sum_{j=1}^{c} b_j(x) \frac{\partial}{\partial x_j}$ near p in these coordinates, and we can compute

<u>exercise</u>: near p
$$[X, Y] = \sum_{j=1}^{c} \sum_{i=1}^{c} \left(a_i \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j} \frac{\partial}{\partial x_i} \right) \in \mathcal{D}$$

$\Rightarrow [X, Y] \in \mathcal{D}$ for any $X, Y \in \mathcal{D}$. ☒.

$\Leftarrow$ next time (sketch).

☒.