Last time: M^m.

- A c-dim'l distribution is a choice of c-dim'l subspace $\mathcal{D}_p \subseteq T_pM$ $\forall p$, smooth if at every p $\exists\, U \ni p$ and (smooth) vector fields $X_1, \ldots, X_c$ spanning $\mathcal{D}$ (at every point in U).

- $\mathcal{D}$ is integrable if near every point $p \in M$ $\mathcal{D} = \mathrm{Span}\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_c}\right)$ for some local coordinates defined near p (i.e., in some chart)

- $\mathcal{D}$ is involutive if whenever $X, Y \in \mathcal{D}$, $[X,Y] \in \mathcal{D}$.

Thm: [Frobenius] A smooth distribution $\mathcal{D}$ (of any dimension) is integrable iff it is involutive.

Proof sketch: $\Rightarrow$ Last time.

$\Leftarrow$ (mostly prove the case of $\dim(\mathcal{D}) = 2$, in $M = \mathbb{R}^3$, leave general case as an exercise).

Say $\mathcal{D}$ is involutive, pick any $p \in M$. Need to find local coordinates $x_1, \ldots, x_m$ with

$$\mathcal{D} = \mathrm{span}\left(\left\{\frac{\partial}{\partial x_i}\right\}_{i=1}^{c}\right).$$

First case:

Prop: Say $X_1, \ldots, X_c$ linearly independent vector fields (over U) with $\mathcal{D} = \mathrm{Span}(X_1, \ldots, X_c)$. Suppose $[X_i, X_j] = 0$ for all i,j. Then $\mathcal{D}$ is integrable (over U).

Pf sketch: (case $c = 2$, $U = \mathbb{R}^3$): Have X, Y linearly independent with $[X,Y] = 0$.

- Can choose coordinates (after possibly shrinking U) s.t. $X = \frac{\partial}{\partial x_1}$ (by earlier in class); in these coordinates $Y = \sum_{i=1}^{3} b_i(x_1,x_2,x_3)\frac{\partial}{\partial x_i}$.

By hypothesis, $[X,Y] = 0$, but

$$[X,Y] = \left[\frac{\partial}{\partial x_1}, \sum b_i(x_1,x_2,x_3)\frac{\partial}{\partial x_i}\right] = \sum_i \left[\frac{\partial}{\partial x_1}, b_i(x_1,x_2,x_3)\frac{\partial}{\partial x_i}\right]$$

(recall that $[A, fB] = A(f)B + f[A,B]$).

$$\sum_i \left(\frac{\partial b_i}{\partial x_1} \frac{\partial}{\partial x_i} + b_i \cancel{\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_i}\right]} \right)$$

0

$$= \sum \frac{\partial b_i}{\partial x_1} \frac{\partial}{\partial x_i} = 0$$

by hypothesis.

(note: recall $[X,Y](f) = X(Y(f)) - Y(X(f))$.

e.g., $\left[\frac{\partial}{\partial x_1}, g\frac{\partial}{\partial x_2}\right](f)$

$$= \frac{\partial}{\partial x_1}\left(g \frac{\partial f}{\partial x_2}\right) - g\frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_1}\right)$$

$$= \frac{\partial g}{\partial x_1}\frac{\partial f}{\partial x_2} + \cancel{g\frac{\partial^2 f}{\partial x_1 \partial x_2}} - \cancel{g\frac{\partial^2 f}{\partial x_2 \partial x_1}}$$

$$= \left(\frac{\partial g}{\partial x_1}\frac{\partial}{\partial x_2}\right)(f)$$

$\Rightarrow$ each b_i satisfies $\frac{\partial b_i}{\partial x_1} = 0$

$\Rightarrow b_i := b_i(x_2, x_3)$

$$\Rightarrow Y = b_1(x_2,x_3)\frac{\partial}{\partial x_1} + b_2(x_2,x_3)\frac{\partial}{\partial x_2} + b_3(x_2,x_3)\frac{\partial}{\partial x_3}.$$

Now, take $Y' = Y - b_1 X = b_2(x_2,x_3)\frac{\partial}{\partial x_2} + b_3(x_2,x_3)\frac{\partial}{\partial x_3}$, only depends on x_2, x_3.

Now, project to $\mathbb{R}^2_{(x_2,x_3)}$, & can integrate Y' to $\frac{\partial}{\partial x_2'}$ & after change of coordinates

↖ new coordinate on $\mathbb{R}^2_{(x_2,x_3)}$

$$\mathcal{D} = \mathrm{Span}(X,Y) = \mathrm{Span}(X, Y') = \mathrm{Span}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2'}\right). \boxtimes$$

General case: Say have $\mathcal{D} = \mathrm{Span}(X_1, \dots, X_c)$ over U &

↖ independent vector fields

$\mathcal{D}$ is involutive..

Again, focus on case $c = 2$, $U = \mathbb{R}^3$; so have X, Y independent

and $[X,Y] \in \mathrm{Span}(X,Y)$, e.g., $[X,Y] \overset{*}{=} AX + BY$ for functions A, B.

(not necessarily 0).

WLOG, pick coordinates s.t. $X = \frac{\partial}{\partial x_1}$ & (by replacing $Y \mapsto Y - \text{something } X$):

$$X = \frac{\partial}{\partial x_1} \text{ and } Y = b_2(x_1,x_2,x_3)\frac{\partial}{\partial x_2} + b_3(x_1,x_2,x_3)\frac{\partial}{\partial x_3}.$$

$$\text{Then } [X,Y] \underset{\text{(exercise: compute)}}{=\!=} \frac{\partial b_2}{\partial x_1}\frac{\partial}{\partial x_2} + \frac{\partial b_3}{\partial x_1}\frac{\partial}{\partial x_3} \overset{*}{=} A\frac{\partial}{\partial x_1} + Bb_2\frac{\partial}{\partial x_2} + Bb_3\frac{\partial}{\partial x_3}.$$

$\Rightarrow A = 0, \quad B b_2 = \frac{\partial b_2}{\partial x_1}, \quad B b_3 = \frac{\partial b_3}{\partial x_1}.$

$$\xRightarrow[\text{for } b_2, b_3]{\text{solve ODE}} \quad b_2 = f(x_2,x_3)\, e^{\int_{t=0}^{t=x_1} B(t,x_2,x_3)\,dt}, \quad b_3 = g(x_2,x_3)\, e^{\int_{t=0}^{t=x_1} B(t,x_2,x_3)\,dt}.$$

$$\Rightarrow Y = e^{\int B}\Big(\underbrace{f(x_2,x_3)\frac{\partial}{\partial x_2} + g(x_2,x_3)\frac{\partial}{\partial x_3}}_{\text{independent of } x_1.}\Big)$$

now, rescale Y to get $Y' = f(x_2,x_3)\frac{\partial}{\partial x_2} + g(x_2,x_3)\frac{\partial}{\partial x_3}$; this is independent of X & $\operatorname{span}(X,Y) = \operatorname{span}(X,Y') = \mathcal{D}$,

& now $[X,Y'] = 0$, reducing to previous case.

(Exercise: general proof). $\boxtimes$.

Lie Groups:

Def: • A Lie group G is a smooth manifold together with smooth maps

$$\mu : G \times G \longrightarrow G \quad \text{and} \quad i : G \longrightarrow G$$

(multiplication) (inverse)

which make G into a group.

• A Lie subgroup is a subgroup $H \subset G$ which is also a submanifold of G.
($\Rightarrow H, \mu|_H, i|_H$ is a Lie group).

Examples:

(1) $GL(n,\mathbb{R}) = \{ A \in M_n(\mathbb{R}) \mid \det(A) \neq 0 \}$. We already showed this is a manifold. The product AB has entries given by polynomials in entries of A & B, hence $(A,B) \longmapsto AB$ is C^∞. Similarly, need to inverse $i : A \longmapsto A^{-1}$ is a C^∞ map ... (exercise).

(2) $SL(n,\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) = 1\}$ is a Lie subgrp of $GL(n,\mathbb{R})$.

(3) $O(n,\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid AA^T = \mathrm{id}\}$ is also a Lie subgrp " ".
$(\Rightarrow \det(A) = \pm 1)$.

(4) $SO(n,\mathbb{R}) = SL(n,\mathbb{R}) \cap O(n,\mathbb{R})$. ("the grp of rigid rotations of $S^{n-1} \subset \mathbb{R}^n$")
$SO(2,\mathbb{R}) = (S^1, *)$.

(5) $\mathbb{R}^n$, $\mu = +$.

(6) $\mathbb{C}^*$, $\mu = *$. (special case of $GL(n,\mathbb{C})$, $n=1$).

(7) $S^1 \subset \mathbb{C}^*$ Lie subgrp ($U(n) \subset GL(n,\mathbb{C})$, $n=1$).
unit circle.

(8) G, G' Lie grps $\rightsquigarrow$ $G \times G'$ is too.

More invariant versions of above examples. V vector space$/\mathbb{R}$.

$\Rightarrow GL(V) = \{\text{invertible } T : V \xrightarrow{\sim} V\}$.

Have $SL(V)$, $O(V, \langle -,- \rangle)$ choice of inner product on V.

Exercises: verify all the above are Lie groups.

A Lie group representation of G is a Lie grp homomorphism (smooth map + grp hom.)

$\phi : G \to GL(V)$.

(ϕ induces an action $G \times V \to V$
$(g,v) \mapsto \phi(g)(v)$.).

Some examples of representations of interest:

(1) $GL(k,\mathbb{R}) \xrightarrow{\det} GL(1,\mathbb{R}) = \mathbb{R}^*$
$A \longmapsto \det(A)$.

(2) $GL(k,\mathbb{R}) \xrightarrow{\text{conj}_B} GL(k,\mathbb{R})$

$$A \longmapsto BAB^{-1}$$

(3) $GL(k,\mathbb{R}) \longrightarrow GL(k,\mathbb{R})$

$$A \longmapsto (A^T)^{-1}.$$

<u>Next time</u>: vector bundles (abstract generalisations of $\begin{array}{c} TM \\ \downarrow \\ M \end{array}$, $\begin{array}{c} T^*M \\ \downarrow \\ M \end{array}$) & use of Lie group representations to construct new vector bundles from old ones.