Vector bundles

$TM \xrightarrow{\pi} M$ and $T^*M \to M$ are first examples of vector bundles.

We'll now set up some general theory in order to define some new vector bundles arising naturally in geometry of manifolds.

Def: A (real, smooth/C^∞) vector bundle of rank k over a manifold M^m is a pair $(E^{m+k}, \pi: E^{m+k} \to M^m)$ satisfying:

smooth $m+k$-dim'l manifold — (surjective) smooth map (a posteriori a submersion)

(linearity of fibers)

(1) $\pi^{-1}(p) =: E_p$ has the structure of a $\mathbb{R}$-vector space of dim. k for each $p \in M$.

(2) ('local triviality') There exists a cover $\{U_\alpha\}$ of M such that $E|_{U_\alpha} := \pi^{-1}(U_\alpha)$ is 'trivializable', meaning that $\exists$ diffeomorphism ('trivialization')

$$\begin{tikzcd}
\pi^{-1}(U_\alpha)=E|_{U_\alpha} \arrow[r, "\sim", "\eta_\alpha"'] \arrow[d, "\pi"] & U_\alpha \times \mathbb{R}^k \arrow[d, "\text{projection}"] \\
U_\alpha \arrow[r, equal] & U_\alpha
\end{tikzcd}$$

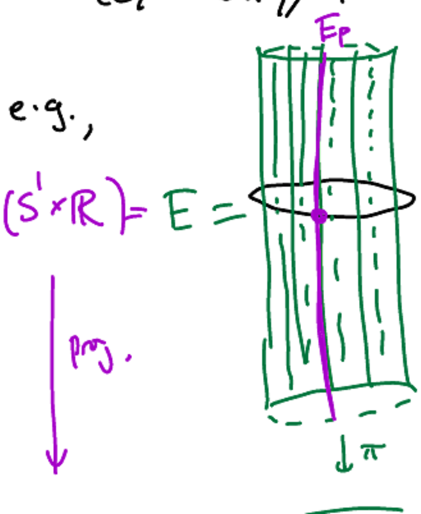
(compatible w/ π)

with $\eta_\alpha|_{E_p=\pi^{-1}(p)} : E_p \xrightarrow{\sim} \{p\}\times\mathbb{R}^k = \mathbb{R}^k$ a linear isomorphism.

(equivalently, for any $p \in M$, $\exists$ nhood $U \ni p$ in M s.t. $E|_U$ is trivializable).

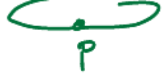
e.g.,

$(S^1\times\mathbb{R}) = E =$



A rank 1 vector bundle is often called a line bundle.

Def: A (real, C^∞) rank ℓ vector sub-bundle of a rank k $E \xrightarrow{\pi} M$ is a submanifold $F \subset E$ such that for each $p \in M$, $F_p := (\pi|_F)^{-1}(p) \subseteq E_p$ linear ℓ-dim'l subspace and $\pi|_F : F \to M$ makes F into a vector bundle over M.

$(S^1 \Rightarrow)$ $M=$ 

Rmk: TM, $\pi: TM \to M^m$ is a rank m vector bundle.

Rmk: $\{$vector sub-bundles of TM of rank $c\}$ $\longleftrightarrow$ $\{$smooth c-dim'l <u>distributions</u> on $M\}$.

(exercise).

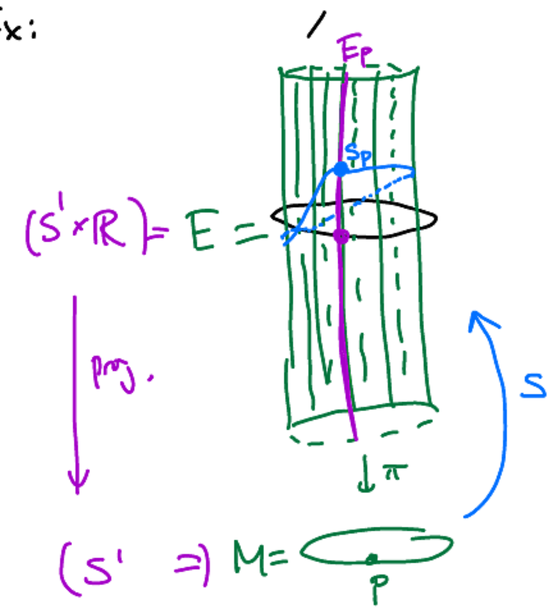
- A <u>section</u> of a vector bundle $E \xrightarrow{\pi} M$ over $U \overset{\text{open}}{\subseteq} M$ is a C^∞ map $s: U \to E$ such that $\underline{\pi \circ s = id_U}$. A section over M is sometimes called a <u>global section</u>.

  $\Rightarrow$ s has the form $p \longmapsto (p, \underset{E_p}{\underset{\cap}{s_p}})$.

Write $\Gamma(E, U) := \{$space of sections of E over $U\}$, with $\Gamma(E) := \Gamma(E, M)$

$\mathbb{R}$-vector space, and also a $C^\infty(M)$-module.

Ex:



<u>Note</u>: $\Gamma(TM) = \mathfrak{X}(M)$ vector fields

$\Gamma(T^*M) = \Omega^1(M)$ one-forms.

Let $\underline{\mathbb{R}^k}: (M \times \mathbb{R}^k, \text{proj}_M: M \times \mathbb{R}^k \to M)$ be the trivial rank k vector bundle.

Note $\Gamma(\underline{\mathbb{R}^{(1)}}) = C^\infty(M)$

$\Gamma(\underline{\mathbb{R}^k}) = C^\infty(M)^k$.

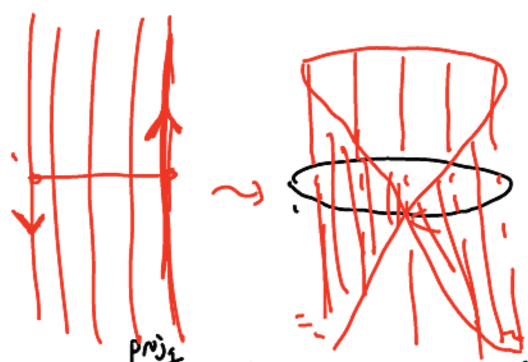
<u>Examples of vector bundles</u>:

- $\underline{\mathbb{R}^k}$ over M for each M.

- Möbius bundle over S^1

  $B \xrightarrow{\pi} S^1$

  $B = [0,1] \times \mathbb{R} \big/ (0,t) \sim (1,-t)$ $\xrightarrow[\pi]{\text{proj}_1}$ $[0,1] \big/ 0 \sim 1 = S^1$



exercise: prove $B \to S^1$ is a vector bundle, but is not globally trivial(izable).

Lemma (1): If $\nexists$ a nowhere-0 section of $E \to M$, then $E \to M$ is not trivial.

(2) E (rank k) is trivial iff $\exists$ a global frame for E, meaning a collection of sections $s_1, \dots, s_k$ with $(s_1)_p, \dots, (s_k)_p$ a basis for E_p for each p.

(E line bundle: $\exists$ a nowhere-0 section).

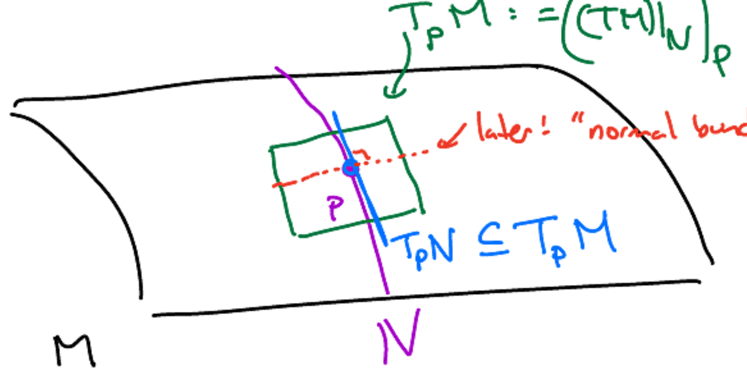
- $TM \xrightarrow{\pi} M$ is a vector bundle (same for $T^*M \to M$ — exercise.)
  - we've constructed a manifold structure on TM, $\pi : TM \to M$ smooth surjective, & $T_pM := \pi^{-1}(p)$ has a vector space structure.
  - local triviality? Fix any $p \in M$. Using a chart (U, ϕ) around p, we showed (by construction) that $\exists$

$$\begin{tikzcd}
TM|_U \arrow[r, "\cong", "(\phi, d\phi)"'] \arrow[d] & \phi(U) \times \mathbb{R}^m = T(\phi(U)) \arrow[r, "(\phi^{-1}, id)", "\cong"'] \arrow[d, "proj."] & U \times \mathbb{R}^m \arrow[d, "proj."] \\
U \arrow[r] & \phi(U) \arrow[r, bend right, "\phi^{-1}"] & U
\end{tikzcd}$$

All together (when applying ϕ^{-1} to RHS as above) we obtain desired local trivialization.

- Given a rank k vector bundle $E \xrightarrow{\pi} M^m$ and a submanifold $N \subseteq M$, we obtain a vector bundle of rank k on N, $E|_N := \pi^{-1}(N)$
$$\begin{array}{c} E|_N \\ \downarrow \pi \\ N. \end{array}$$

Ex: $N^n \subseteq M^m$, can consider $(TM)|_N$, rank m bundle, & note $\underset{\text{rank } n}{TN} \underset{\text{sub-bundle}}{\subseteq} (TM)|_N$



later! "normal bundle to N in M as vec. bdle quotient" $\left(TM|_N\right) / TN$.

- More generally, if $f: Q^q \longrightarrow M^m$ any C^∞ map, and $\begin{matrix} E \\ \downarrow \pi \\ M \end{matrix}$ rank k vector bundle, we can define the <u>pullback of E along f</u>

$$\begin{matrix} f^*E := \{(q,e) \in Q \times E \mid \underset{M}{\pi(e)} = f(q)\}. \\ \downarrow \\ Q \end{matrix} \qquad (f^*E)_q = E_{f(q)}.$$

(if $Q \overset{i}{\hookrightarrow} M$ submanifold then $i^*E = E|_Q$.)

<u>Next time</u>: Many vector space operations extend to the world of vector bundles, giving ways to construct new vec. bundles from old ones.

(e.g., $V \oplus W$, V/W, $V \otimes W$, $\Lambda^k V$, $\mathrm{Hom}(V,W)$, ...)

apply to T^*M; sections of $\Lambda^k T^*M$ are <u>differential k-forms</u>