Last time:

Def: A (real, smooth/C^∞) vector bundle of rank k over a manifold M^m is a pair $(E^{m+k}, \pi: E^{m+k} \to M^m)$ satisfying:

smooth $m+k$-dim'l manifold

(surjective) smooth map (a posteriori a submersion)

(linearity of fibers)

(1) $\pi^{-1}(p) =: E_p$ has the structure of a $\mathbb{R}$-vector space of dim. k for each $p \in M$.

(2) ('local triviality') There exists a cover $\{U_\alpha\}$ of M such that $E|_{U_\alpha} := \pi^{-1}(U_\alpha)$ is 'trivializable,' meaning that $\exists$ diffeomorphism ('trivialization')

$$\begin{tikzcd} \pi^{-1}(U_\alpha) = E|_{U_\alpha} \arrow[r, "\sim", "\tau_\alpha"'] \arrow[d, "\pi"] & U_\alpha \times \mathbb{R}^k \arrow[d, "\text{projection}"] \\ U_\alpha \arrow[r, equal] & U_\alpha \end{tikzcd}$$

(compatible w/ π)

with $\tau_\alpha|_{E_p = \pi^{-1}(p)} : E_p \xrightarrow{\sim} \{p\} \times \mathbb{R}^k = \mathbb{R}^k$ a linear isomorphism.

- A (global) section of $(E, \pi: E \to M)$ is a (smooth) map $s: M \to E$ w/ $\pi \circ s = \mathrm{id}_M$

the vector bundle $E|_U = \pi^{-1}(U) \to U$

- set of sections forms a vector space, denoted $\Gamma(E)$ $\quad (\Gamma(U; E) := \Gamma(E|_U))$

- $\Gamma(TM) = \mathfrak{X}(M)$ vector field, $\Gamma(T^*M) = \Omega^1(M)$, $\Gamma(\underline{\mathbb{R}}) = C^\infty(M)$

Transition functions / 'gluing data'.

Given local trivializations of E:

- over U:

$$\begin{tikzcd} E|_U \arrow[r, "\Phi_U", "\cong"'] \arrow[d, "\pi"] & U \times \mathbb{R}^k \arrow[d, "\text{proj.}"] \\ U \arrow[r, equal] & U \end{tikzcd}$$

- over V:

$$\begin{tikzcd}
E|_V \arrow[r, "\Phi_V", "\cong"'] \arrow[d, "\pi"] & V \times \mathbb{R}^k \arrow[d, "\text{proj.}"] \\
V \arrow[r, phantom, "="] & V
\end{tikzcd}$$

If $U \cap V \neq \emptyset$, we get a transition function associated to this pair of trivializations:

$$\begin{tikzcd}
(p,v) \arrow[d, mapsto] & (U\cap V) \times \mathbb{R}^k \arrow[r, "(\Phi_U)^{-1}", "\cong"'] \arrow[rr, bend right, "\overline{\Phi}_{UV}"'] \arrow[dr, "\text{proj}"'] & E|_{U \cap V} \arrow[r, "\Phi_V", "\cong"'] & (U \cap V) \times \mathbb{R}^k \arrow[dl, "\text{proj}"] & (p,v) \arrow[d, mapsto] \\
p & & U \cap V & & p
\end{tikzcd}$$

Compatibility with projection tells us that $\overline{\Phi}_{UV}(p,v) = (p, \Phi_{UV}(p)(v))$

where $\Phi_{UV} : U \cap V \longrightarrow GL(k, \mathbb{R})$ a smooth map.

In other words, we can think of the associated vector bundle transition function

as $\Phi_{UV} : U \cap V \longrightarrow GL(k, \mathbb{R})$.

If $\{(U_\alpha, \Phi_\alpha)\}_{\alpha \in I}$ is a trivializing cover for E, meaning Φ_α trivializes $E|_{U_\alpha}$, & U_α cover M,

$\rightsquigarrow$ transition functions $\Phi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(k, \mathbb{R})$ for each $\alpha, \beta \in I$.

Satisfying:

(1) $\Phi_{\alpha\beta} \cdot \Phi_{\beta\alpha} = \underline{id} : U_\alpha \cap U_\beta \longrightarrow GL(k, \mathbb{R})$.

$$p \longmapsto \Phi_{\alpha\beta}(p) \cdot \Phi_{\beta\alpha}(p) = Id_{k \times k}.$$

(equivalently, $\overline{\Phi}_{\alpha\beta} \circ \overline{\Phi}_{\beta\alpha} = id_{(U_\alpha \cap U_\beta \times \mathbb{R}^k)}$).

(2) For any α, β, γ

$$\Phi_{\alpha\gamma} = \Phi_{\beta\gamma} \cdot \Phi_{\alpha\beta}$$

as maps $U_\alpha \cap U_\beta \cap U_\gamma \longrightarrow GL(k, \mathbb{R})$.

(cocycle condition)

Prop: Conversely, given an open cover $\{U_\alpha\}_{\alpha \in I}$ of M & (smooth) functions

$$\Phi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(k, \mathbb{R}) \quad \forall \ \alpha, \beta \in I.$$

satisfying:

(1) $\Phi_{\alpha\beta} \cdot \Phi_{\beta\alpha} = Id_{k \times k}$

(2) cocycle condition: on $U_\alpha \cap U_\beta \cap U_\gamma$, $\Phi_{\alpha\gamma} = \Phi_{\beta\gamma} \cdot \Phi_{\alpha\beta}$.

Then, one can construct a vector bundle

$E \xrightarrow{\pi} M$ where $\Phi_{\alpha\beta}$ the "change of trivialization" transition functions,

roughly by gluing together trivial bundles $\underline{\mathbb{R}}^k_{U_\alpha} = U_\alpha \times \mathbb{R}^k$ along the $\Phi_{\alpha\beta}$'s.

Specifically,

$$E = \coprod_{\alpha \in I} U_\alpha \times \mathbb{R}^k \Big/ \begin{array}{l} \forall \ \alpha, \beta \text{ and } \forall \ p \in U_\alpha \cap U_\beta \\ \underset{\in U_\alpha \times \mathbb{R}^k}{(p, v)} \sim \underset{\in U_\beta \times \mathbb{R}^k}{(p, \Phi_{\alpha\beta}(p)(v))}. \end{array}$$

$$\Big\downarrow \pi$$

$$M \underset{\text{check}}{=} \coprod_{\alpha \in I} U_\alpha \Big/ \begin{array}{l} \forall \ \alpha, \beta \text{ and } p \in U_\alpha \cap U_\beta \\ \underset{\in U_\alpha}{p} \sim \underset{\in U_\beta}{p}. \end{array}$$

(note: in particular, if each U_α was part of a chart (U_α, ϕ_α), we could similarly view M as being built by gluing together open subsets of Euclidean space $\phi_\alpha(U_\alpha)$ along $\phi_\beta \circ \phi_\alpha^{-1}$ maps.)

To check: The resulting E is a vector bundle over M satisfying the proposition (exercise).

$$\coprod_{\alpha \in I} U_\alpha := \left\{ (\alpha, p) \;\middle|\; \begin{matrix} \alpha \in I \\ p \in U_\alpha \end{matrix} \right\}.$$

In fact, $\left\{ \begin{matrix} \text{rank } k\text{-vector bundles} \\ \text{on } M \end{matrix} \right\} \Big/ \text{isomorphism} \xleftrightarrow{1:1} \left\{ \text{gluing data as above} \right\} \Big/$ some equiv. relation, (which we haven't described).

A map of vector bundles over M is a smooth f as below

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi_E"'] & \circlearrowright & F \arrow[dl, "\pi_F"] \\ & M & \end{tikzcd}$$

s.t., $f_p : E_p \to F_p$ is a linear map.

Isomorphism: each f_p is an iso.

Sometimes, we look at maps of vector bundles over maps of spaces

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi_E"'] \arrow[dr, phantom, "\circlearrowright"] & F \arrow[d, "\pi_F"] \\ M \arrow[r, "g"'] & N \end{tikzcd}$$

(where $f_p : E_p \to F_{g(p)}$ is linear for all p.)

(e.g., $g : M \to N$ any C^∞ map then dg induces such a map

$$\begin{tikzcd} TM \arrow[r, "dg"] \arrow[d, "\pi"'] & TN \arrow[d, "\pi"] \\ M \arrow[r, "g"] & N \end{tikzcd}$$

).

(case of $\begin{array}{ccc} E & \to & F \\ & \searrow \swarrow & \\ & M & \end{array}$ is the special case $\begin{array}{ccc} E & \to & F \\ \downarrow & & \downarrow \\ M & \underset{=}{\xrightarrow{id}} & M \end{array}$).

Constructing new vector bundles from old

Say E is a vector bundle over M, of rank k.

(1) via representations of $GL(k,\mathbb{R})$:

Given a representation $\rho: GL(k,\mathbb{R}) \to GL(\ell,\mathbb{R})$,

ex: (i) $GL(k,\mathbb{R}) \to GL(1,\mathbb{R})$

$A \longmapsto \det(A)$

(ii) $GL(k,\mathbb{R}) \to GL(k,\mathbb{R})$

$A \longmapsto BAB^{-1}$

(iii) $GL(k,\mathbb{R}) \to GL(k,\mathbb{R})$

$A \longmapsto (A^T)^{-1}$.

and a collection of transition functions $\Phi_{\alpha\beta}: U_\alpha \cap U_\beta \to GL(k,\mathbb{R})$ for E

(associated to an open cover $\{U_\alpha\}_{\alpha\in I}$ of M w/ trivializations Φ_α of $E(U_\alpha)$).

Using the representation $\rho: GL(k,\mathbb{R}) \to GL(\ell,\mathbb{R})$, we get

$$\Psi_{\alpha\beta} := \rho \circ \Phi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(\ell,\mathbb{R}).$$

(Prop above)

$\rightsquigarrow$ A new rank ℓ vector bundle, which we call $\underbrace{E \times_\rho \mathbb{R}^\ell}$ "E twisted by ρ".

Ex: Representation (i) (det) applied to $E = T^*M$ gives a line bundle

$$T^*M \times_{\det} \mathbb{R} =: \Lambda^n T^*M,$$ which we'll define another (more direct) way.

Representation (iii) $\left(A \mapsto (A^T)^{-1}\right)$ applied to $E = TM$

produces T^*M.

Next time: another perspective on such operations, & linear algebra of creating new vector bundles from old (goal: define $\Lambda^k T^*M$ for every k).