

Last time:

Def: A (real, smooth/ C^∞) vector bundle of rank k over a manifold M^n is a pair $(E^{m+k}, \pi: E^{m+k} \rightarrow M^n)$ satisfying:

smooth $m+k$ -dim manifold (pointing to E^{m+k})
 (surjective) smooth map (a posteriori a submersion) (pointing to π)

(linearity of fibers)

(1) $\pi^{-1}(p) =: E_p$ has the structure of a \mathbb{R} -vector space of dim. k for each $p \in M$.

(2) ('local triviality') There exists a cover $\{U_\alpha\}$ of M such that $E|_{U_\alpha} =: \pi^{-1}(U_\alpha)$ is 'trivializable', meaning that \exists diffeomorphism ('trivialization')

$$\pi^{-1}(U_\alpha) = E|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times \mathbb{R}^k$$

(compatible w/ π)

$$\begin{array}{ccc} \downarrow \pi & \dashrightarrow & \downarrow \text{projection} \\ U_\alpha & \cong & U_\alpha \end{array}$$

with $\pi_\alpha|_{E_p = \pi^{-1}(p)}: E_p \xrightarrow{\cong} \{p\} \times \mathbb{R}^k = \mathbb{R}^k$ a linear isomorphism.

- A section (global) of $(E, \pi: E \rightarrow M)$ is a smooth map $s: M \rightarrow E$ w/ $\pi \circ s = \text{id}_M$ the vector bundle $E|_{U_\alpha = \pi^{-1}(U_\alpha)} \rightarrow U_\alpha$
- set of sections forms a vector space, denoted $\Gamma(E)$ ($\Gamma(U; E) =: \Gamma(E|_U)$)

• $\Gamma(TM) = \mathfrak{X}(M)$ vector field, $\Gamma(T^*M) = \Omega^1(M)$, $\Gamma(\underline{\mathbb{R}}) = C^\infty(M)$

Transition functions / 'gluing data'

Given local trivializations of E :

• over U :

$$\begin{array}{ccc} E|_U & \xrightarrow[\cong]{\Phi_U} & U \times \mathbb{R}^k \\ \downarrow \pi & & \downarrow \text{proj.} \\ U & = & U \end{array}$$

$$\begin{array}{ccc} \bullet \text{ over } V : & E|_V & \xrightarrow[\cong]{\Phi_V} & V \times \mathbb{R}^k \\ & \downarrow \pi & & \downarrow \text{proj.} \\ & V & = & V \end{array}$$

If $U \cap V \neq \emptyset$, we get a transition function associated to this pair of trivializations:

$$\begin{array}{ccccc} (p,v) & (U \cap V) \times \mathbb{R}^k & \xrightarrow[\cong]{(\Phi_U)^{-1}} & E|_{U \cap V} & \xrightarrow[\cong]{\Phi_V} & (U \cap V) \times \mathbb{R}^k & (p,v) \\ & \searrow & & \text{proj.} & \searrow & & \\ & & & U \cap V & & & \\ & & & \text{proj.} & & & \end{array}$$

$\overline{\Phi}_{UV}$

Compatibility with projection tells us that $\overline{\Phi}_{UV}(p,v) = (p, \Phi_{UV}(p)(v))$
 where $\Phi_{UV} : U \cap V \rightarrow GL(k, \mathbb{R})$ a smooth map.

In other words, we can think of the associated vector bundle transition function

$$\text{as } \overline{\Phi}_{UV} : U \cap V \rightarrow GL(k, \mathbb{R}).$$

If $\{(U_\alpha, \Phi_\alpha)\}_{\alpha \in I}$ is a trivializing cover for E , meaning Φ_α trivializes $E|_{U_\alpha}$,
 $\mathcal{B} U_\alpha$ cover M_r

\leadsto transition functions $\overline{\Phi}_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(k, \mathbb{R})$ for each $\alpha, \beta \in I$.

Satisfying:

$$(1) \overline{\Phi}_{\alpha\beta} \cdot \overline{\Phi}_{\beta\alpha} = \text{id} : U_\alpha \cap U_\beta \rightarrow GL(k, \mathbb{R}).$$

$$p \longmapsto \overline{\Phi}_{\alpha\beta}(p) \cdot \overline{\Phi}_{\beta\alpha}(p) = \text{Id}_{k \times k}.$$

(equivalently, $\overline{\Phi}_{\alpha\beta} \cdot \overline{\Phi}_{\beta\alpha} = \text{id}_{(U_\alpha \cap U_\beta) \times \mathbb{R}^k}$.)

(2) For any α, β, γ

$$\Phi_{\alpha\gamma} = \Phi_{\beta\gamma} \circ \Phi_{\alpha\beta}$$

as maps $U_\alpha \cap U_\beta \cap U_\gamma \rightarrow GL(k, \mathbb{R})$.

(cocycle condition)

Prop: Conversely, given an open cover $\{U_\alpha\}_{\alpha \in I}$ of M & (smooth) functions

$$\Phi_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow GL(k, \mathbb{R}) \quad \forall \alpha, \beta \in I.$$

satisfying:

$$(1) \Phi_{\alpha\beta} \circ \Phi_{\beta\alpha} = \underline{I}_{k \times k}$$

$$(2) \text{ cocycle condition: on } U_\alpha \cap U_\beta \cap U_\gamma, \Phi_{\alpha\gamma} = \Phi_{\beta\gamma} \circ \Phi_{\alpha\beta}.$$

Then, one can construct a vector bundle

$E \xrightarrow{\pi} M$ where $\Phi_{\alpha\beta}$ the "change of trivialization" transition functions,

roughly by gluing together trivial bundles $\mathbb{R}^k_{U_\alpha} = U_\alpha \times \mathbb{R}^k$ along $\Phi_{\alpha\beta}$'s.

Specifically,

$$E = \bigsqcup_{\alpha \in I} U_\alpha \times \mathbb{R}^k \Big/ \begin{array}{l} \forall \alpha, \beta \text{ and } \forall p \in U_\alpha \cap U_\beta \\ (p, v) \sim (p, \Phi_{\alpha\beta}(p)(v)) \\ \uparrow \qquad \qquad \qquad \uparrow \\ U_\alpha \times \mathbb{R}^k \qquad \qquad U_\beta \times \mathbb{R}^k \end{array}$$

$$M \xlongequal{\text{check}} \bigsqcup_{\alpha \in I} U_\alpha \Big/ \begin{array}{l} \forall \alpha, \beta \text{ and } p \in U_\alpha \cap U_\beta \\ p \sim p \\ \uparrow \qquad \qquad \qquad \uparrow \\ U_\alpha \qquad \qquad \qquad U_\beta \end{array}$$

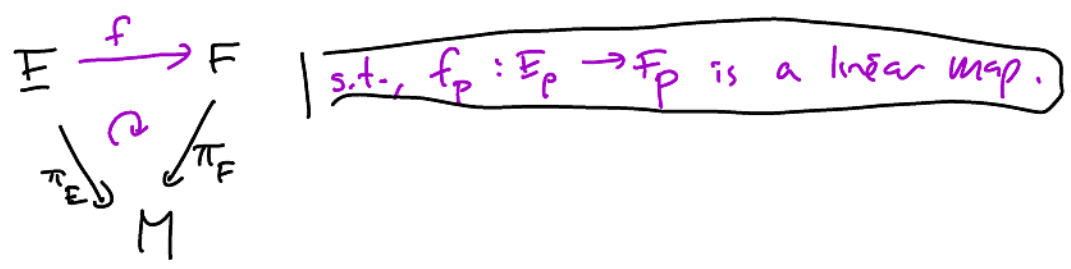
note: in particular, if each U_α was part of a chart (U_α, ϕ_α) , we could similarly view M as being built by gluing together open subsets of Euclidean space $\phi_\alpha(U_\alpha)$ along $\phi_\beta \circ \phi_\alpha^{-1}$ maps.

To check: The resulting E is a vector bundle over M satisfying the proposition (exercise).

$$\coprod_{\alpha \in I} U_\alpha := \left\{ (\alpha, p) \mid \begin{array}{l} \alpha \in I \\ p \in U_\alpha \end{array} \right\}$$

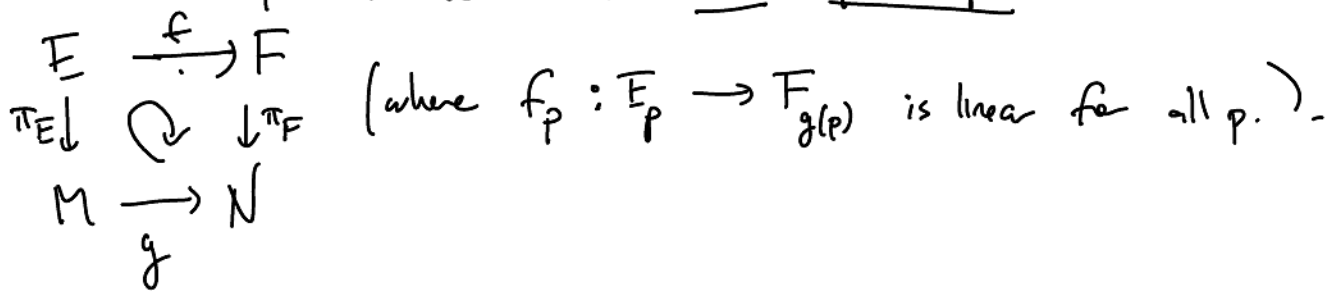
In fact, $\left\{ \begin{array}{l} \text{rank } k\text{-vector bundles} \\ \text{on } M \end{array} \right\} \xrightarrow{1:1} \left\{ \begin{array}{l} \text{gluing data as above} \end{array} \right\}$
 isomorphism. / some equiv. relation, (which we haven't described).

A map of vector bundles over M is a smooth f as below

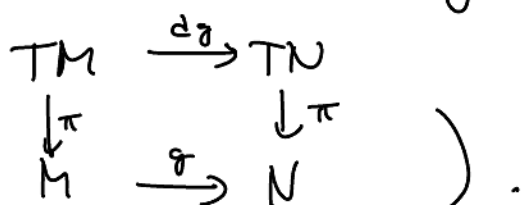


isomorphism: each f_p is an iso.

Sometimes, we look at maps of vector bundles over maps of spaces



(e.g., $g: M \rightarrow N$ any C^∞ map then dg induces such a map



(case of $\begin{array}{ccc} E & \rightarrow & F \\ \downarrow & & \downarrow \\ & M & \end{array}$ is the special case $\begin{array}{ccc} E & \rightarrow & F \\ \downarrow & & \downarrow \\ M & \xrightarrow{\text{id}} & M \end{array}$).

Constructing new vector bundles from old

Say E is a vector bundle over M , of rank k .

(1) via representations of $GL(k, \mathbb{R})$:

Given a representation $\rho: GL(k, \mathbb{R}) \rightarrow GL(\ell, \mathbb{R})$,

ex: (i) $GL(k, \mathbb{R}) \rightarrow GL(1, \mathbb{R})$

$$A \longmapsto \det(A)$$

(ii) $GL(k, \mathbb{R}) \rightarrow GL(k, \mathbb{R})$

$$A \longmapsto BAB^{-1}$$

(iii) $GL(k, \mathbb{R}) \rightarrow GL(k, \mathbb{R})$

$$A \longmapsto (A^T)^{-1}$$

and a collection of transition functions $\Phi_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow GL(k, \mathbb{R})$ for E

(associated to an open cover $\{U_\alpha\}_{\alpha \in I}$ of M w/ transition functions Φ_α of $E|_{U_\alpha}$).

Using the representation $\rho: GL(k, \mathbb{R}) \rightarrow GL(\ell, \mathbb{R})$, we get

$$\Psi_{\alpha\beta} := \rho \circ \Phi_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow GL(\ell, \mathbb{R}).$$

(Prop above)

\Rightarrow A new rank ℓ vector bundle, which we call $\underbrace{E \times \mathbb{R}^\ell}_\rho$ " E twisted by ρ ".

Ex: Representation (i) (\det) applied to $E = T^*M$ gives a line bundle

$$T^*M \times_{\det} \mathbb{R} =: \Lambda^m T^*M, \text{ which we'll define another (more direct) way.}$$

Representation (ii) $(A \mapsto (A^T)^{-1})$ applied to $E = TM$

produces T^*M .

Next time: another perspective on such operators, & linear algebra of creating new vector bundles from old (goal: define $\Lambda^k T^*M$ for every k).