Vector bundles

How to construct new vector bundles from old?

From last times:

(1) $N \subset M$, $E \xrightarrow{\pi} M$ (rank k), then $E|_N \xrightarrow{\pi} N$ is a rank k vector bundle on N.

(2) More generally $\begin{matrix} E \\ \pi \downarrow \\ M \end{matrix}$ and $g: Q \to M$ any C^∞ map $\rightsquigarrow$ $\left(\begin{matrix} g^*E \\ \downarrow g^*\pi \\ Q \end{matrix}\right)$.

$(g^*E)_q = E_{g(q)}$

(3) Given a rank $\underline{k}$ vec. bundle $\begin{matrix} E \\ \downarrow \pi \\ M \end{matrix}$, Lie group rep. $\rho: GL(\underline{k}) \to GL(\ell)$,

we get a new vector bundle $\begin{matrix} E \times_\rho \mathbb{R}^\ell \\ \downarrow \\ M \end{matrix}$.

(4) Many vector space operations extend to the world of vector bundles

($\oplus$, $\otimes$, $\mathrm{Hom}_{\mathbb{R}}(-,-)$, quotient)

Ex: ($\oplus$ of vector bundles):

(i) If $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$, $\begin{matrix} F \\ \downarrow \\ N \end{matrix}$ vec. bundles, then $\begin{matrix} E \times F \\ \downarrow (\pi_E, \pi_F) \\ M \times N \end{matrix}$.

(ii) Given $\begin{matrix} E, F \\ \downarrow \\ M \end{matrix}$, define $\begin{matrix} E \oplus F \\ \downarrow \\ M \end{matrix} := \Delta^* \left(\begin{matrix} E \times F \\ \downarrow \\ M \times M \end{matrix} \right)$ $\quad \Delta: M \to M \times M$, $p \mapsto (p,p)$

"Whitney sum of vector bundles"

$$\Delta^*(E \times F)_q := (E \times F)_{\Delta(q)} = (E \times F)_{(q,q)}$$
$$= E_q \times F_q$$
$$= E_q \oplus F_q .$$

Ex: 2: (exercise)

If $\begin{matrix} E_1 \subseteq E_2 \\ {}_{\pi_{E_1}} \searrow \ \swarrow {}_{\pi_{E_2}} \\ M \end{matrix}$ vector sub-bundle, then $\exists$ a vector bundle $\begin{matrix} E_2/E_1 \\ \downarrow \pi_{E_2/E_1} \\ M \end{matrix}$

with fibers $(E_2/E_1)_p = (E_2)_p/(E_1)_p$.

Ex. 3: (exercise): If $\begin{array}{c}E\\ \downarrow\\ M\end{array}$, then $\exists$ vector bundle $\begin{array}{c}E^*\\ \downarrow\\ M\end{array}$ with $(E^*)_p = (E_p)^*$.

Ex. 3' (exercise): If $\begin{array}{c}E_1\\ \downarrow\\ M\end{array}, \begin{array}{c}E_2\\ \downarrow\\ M\end{array}$ vector bundles over M, then $\exists$ vector bundle

$\begin{array}{c}\underline{\mathrm{Hom}}(E_1,E_2)\\ \downarrow\\ M\end{array}$ with fibers $\underline{\mathrm{Hom}}(E_1,E_2)_p = \mathrm{Hom}_{\mathbb{R}}((E_1)_p, (E_2)_p)$.

To construct more such operations on vector bundles, let's recall/define tensor & wedge products of vector spaces.

Linear algebra of tensor products

V, W vector spaces over $\mathbb{R}$. Some previously defined vector spaces built from V, W are:

(a) direct sum $V \oplus W$. As a set $V \oplus W = V \times W$, addition $(v_1,w_1)+(v_2,w_2) = (v_1+v_2, w_1+w_2)$, $c(v_1,w_1) = (cv_1, cw_1)$. $\dim(V\oplus W) = \dim(V) + \dim(W)$.

covariant in V and W.

(b) $\mathrm{Hom}_{\mathbb{R}}(V,W) = \{\mathbb{R}\text{-linear maps } V \to W\}$. $\qquad V^* := \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$

$\dim(\mathrm{Hom}_{\mathbb{R}}(V,W)) = (\dim V)\cdot(\dim W)$

covariant in W, contravariant in V (meaning $f: V \to V'$ induces
$f^*: \mathrm{Hom}_{\mathbb{R}}(V',W) \longrightarrow \mathrm{Hom}(V,W)$
$T \longmapsto T\circ f$
opposite direction)

Tensor product $V \otimes W$

- $\dim(V\otimes W) = (\dim V)\cdot(\dim W)$ (like $\mathrm{Hom}_{\mathbb{R}}(V,W)$)
- covariant in V and W. (like $V \oplus W$)

To define: recall that a map $V \times W \xrightarrow{\phi} Z$ is bilinear if

$$\phi(av_1 + bv_2, w) = a\,\phi(v_1,w) + b\,\phi(v_2,w)$$

and $\phi(v, cw_1 + dw_2) = c\,\phi(v,w_1) + d\,\phi(v,w_2)$.

Def: (inexplicit) The tensor product $V \otimes_{(\mathbb{R})} W$ is an $\mathbb{R}$-vector space equipped with a bilinear map $\phi : V \times W \to V \otimes W$ which satisfies the following <u>universal property</u>:

notation: $\phi(v,w) =: v \otimes w$.

(meaning this property characterizes $V\otimes W$ up to <u>unique</u> isomorphism; any other $V\otimes W$ satisfying same property is uniquely iso. to $V\otimes W$).

If $\eta : V \times W \to Z$ any other bilinear map, then there exists a <u>unique</u> factorization

$$\begin{tikzcd} V\times W \arrow[r, "\phi"] \arrow[rr, bend right, "\eta"'] & V\otimes W \arrow[r, dashed, "\overline{\eta}"] & Z \end{tikzcd}$$

, where $\overline{\eta}$ is a linear map.

<u>Exercise</u>: verify that if $(V\otimes W)_1$ and $(V\otimes W)_2$ both satisfy above property for ϕ_1, ϕ_2, then $\exists !$ iso. $(V\otimes W)_1 \xrightarrow{\cong} (V\otimes W)_2$.

$$\begin{tikzcd} (V\otimes W)_1 \arrow[rr, "\cong"] & & (V\otimes W)_2 \\ & V\times W \arrow[ul, "\phi_1"] \arrow[ur, "\phi_2"'] \arrow[u, phantom, "\circlearrowright"] & \end{tikzcd}$$

<u>Rmks:</u>

(a) For every $v \in V$, $w \in W$ $\exists$ an element $v \otimes w := \phi(v,w) \in V \otimes W$. Call such elements of $V\otimes W$ <u>pure tensors</u>; general elements of $V \otimes W$ need not be pure tensors, but must be sums of pure tensors.

(b) The above univ. property in particular implies that $\exists$ iso. of vector spaces (for any Z)

$$\mathrm{Hom}_{\mathbb{R}}(V\otimes W, Z) \cong \mathrm{Bilinear\,Hom}_{\mathbb{R}}(V \times W, Z)$$

$$\overline{\eta} : V\otimes W \to Z \quad \longmapsto \quad \overline{\eta}\circ\phi : V\times W \to Z.$$

The univ. property above guarantees uniqueness of $(V\otimes W, \phi)$, but not existence.

<u>One actual existence construction</u>: (for $V\otimes W$)

If S any set, denote by $F_{\mathbb{R}}(S)$ the free $\mathbb{R}$-vec. space generated by S.

, finite sums,

$$F_{\mathbb{R}}(S) = \left\{ \sum_{i=1}^{k} a_i s_i \;\middle|\; \begin{array}{l} a_i \in \mathbb{R} \\ s_i \in S \end{array} \right\}. \left(= \bigoplus_{s \in S} \underbrace{\mathbb{R}\langle s \rangle} \right.$$

means $\mathbb{R}$, but in $\oplus$ we'll call an element a in this copy of $\mathbb{R}$ $a \cdot s$.

(e.g., $F_{\mathbb{R}}(\{1, \ldots, k\}) \cong \mathbb{R}^k$)

We will begin with $V \times W$ as a set, indicating elements by (v,w), and first take

$$F_{\mathbb{R}}(V \times W) = \left\{ \sum_{\text{finite}} \overset{\mathbb{R}}{a_{ij}} (\overset{V}{v_i}, \overset{W}{w_j}) \right\}.$$

There's a map $V \times W \xrightarrow{\phi} F_{\mathbb{R}}(V \times W)$

$$v, w \longmapsto \underline{1}(v,w).$$

but this isn't a bilinear map, and hence $F_{\mathbb{R}}(V \times W)$ is not quite $V \otimes W$, because we'd want certain relations to hold in $V \otimes W$ to make ϕ bilinear.

e.g., we'd want

$$\phi(v_1 + v_2, w) = \phi(v_1, w) + \phi(v_2, w)$$

etc.

We consider the vector subspace $R(V,W) \subset F_{\mathbb{R}}(V \times W)$ spanned by <u>bilinear relations</u>

$$R(V,W) = \operatorname{Span} \left\{ \begin{array}{l} (v_1 + v_2, w) - (v_1, w) - (v_2, w), \\ (v, w_1 + w_2) - (v, w_1) - (v, w_2) \\ (cv, w) - c(v, w) \\ (v, cw) - c(v, w) \end{array} \;\middle|\; \begin{array}{c} v, w, c, \\ v_1, v_2, w_1, w_2 \\ \text{arbitrary} \end{array} \right\}$$

and let $V \otimes W := F_{\mathbb{R}}(V,W) / R(V,W)$.

The map $\phi : V \times W \to F_{\mathbb{R}}(V \times W)$ descends to a map, we'll also call ϕ:

$$\begin{tikzcd} V \times W \arrow[r] \arrow[rr, bend right, "\phi"'] & F_{\mathbb{R}}(V \times W) \arrow[r, "\text{quotient}"] & V \otimes W \end{tikzcd}$$

<u>Exercise</u>: verify ϕ is bilinear and $(V \otimes W, \phi)$ satisfy the universal property of the definition.

Again, refer to $\phi(v,w) = [\underline{1}(v,w)]$ as $\boxed{v \otimes w}$ <u>pure tensor</u>

<u>Key property</u>:

$\exists$ canonical linear map $V^* \otimes W \xrightarrow{\alpha} \mathrm{Hom}_{\mathbb{R}}(V,W)$, which is an isomorphism if V, W finite-dimensional.

To construct linear maps out of $V^* \otimes W$, it suffices by univ. property to construct a bilinear map $V^* \times W \to \mathrm{Hom}_{\mathbb{R}}(V,W)$

(& appeal to $\mathrm{BilinearMap}(V^* \times W, Z) \cong \mathrm{Hom}_{\mathbb{R}}(V^* \times W, Z) \quad \forall Z.$).

$$\bar{\eta} \circ \phi \longleftarrow\!\!\shortmid \bar{\eta}$$

$$\begin{aligned} V^* \times W &\longrightarrow \mathrm{Hom}_{\mathbb{R}}(V,W) \\ (\phi, w) &\longmapsto T_{\phi,w} : \underset{\substack{\in \\ V}}{v} \longmapsto \underset{\substack{\in \\ \mathbb{R}.}}{\phi(v)}\, w \end{aligned}$$

check: bilinear in (ϕ, w).