Last time: Constructed, for V, W vec. spaces/$\mathbb{R}$,

the tensor product $V \underset{\mathbb{R}}{\otimes} W$, comes equipped w/ bilinear map $\phi : V \times W \to V \otimes W$

$$(v,w) \longmapsto \phi(v,w) \overset{\shortparallel}{} v \otimes w$$

Universal property: For any bilinear ψ as in diagram:

$$\begin{tikzcd}
V \times W \arrow[r, "\psi"] \arrow[d, "\phi"'] & Z \\
V \otimes W \arrow[ur, dashed, "\exists ! \overline{\psi}"'] &
\end{tikzcd}$$

$\exists !$ $\overline{\psi} : V \otimes W \to Z$ linear map as in diagram making diagram commute.

Properties:

(1) If $\{v_1, \dots, v_k\}$ basis for V, $\{w_1, \dots, w_\ell\}$ basis for W

$\Rightarrow \{v_i \otimes w_j\}_{i,j}$ is a basis for $V \otimes W$

$\Rightarrow \dim(V \otimes W) = \dim(V) \cdot \dim(W)$.

(2) $\exists$ canonical map $V^* \otimes W \xrightarrow[\cong]{\alpha} \mathrm{Hom}_{\mathbb{R}}(V,W)$

induced by the bilinear map $\nwarrow$ iso. when V, W finite-dim'l.

$$V^* \times W \to \mathrm{Hom}_{\mathbb{R}}(V,W)$$

$$\phi \quad w \longmapsto T_{\phi,w} : v \longmapsto \phi(v) w .$$

α is iso. when V, W finite-dim'l (check: sends a basis to a basis (exercise)).

(3) $V \otimes \mathbb{R} \xrightarrow{\cong} V$

(4) $V \otimes W \xrightarrow{\sim} W \otimes V$.

(how to check this? need to produce a linear map $V \otimes W \to W \otimes V$, can get this from a bilinear map $V \times W \to W \otimes V$. The above map is induced by

bilinear map $\quad V \times W \underset{\text{swap}}{\longrightarrow} W \times V \xrightarrow{\phi} W \otimes V$

Then construct two-sided inverse $W \otimes V \to V \otimes W$ in same way).

(5) $V\otimes(W\oplus Z)\xrightarrow{\cong}(V\otimes W)\oplus(V\otimes Z)$

canonical isomorphisms.

(6) $V\otimes(W\otimes Z)\xrightarrow{\cong}(V\otimes W)\otimes Z$.

<u>Tensor</u> <u>algebra associated to V</u> :

convention: $V^{\otimes 0}=R$

Given $V/_{R}$, define $T(V):=\mathbb{R}\oplus V\oplus V^{\otimes 2}\oplus V^{\otimes 3}\oplus\cdots$

$$=\bigoplus_{k\geqslant 0}V^{\otimes k}\,.\ \ni\sum_{i=0}^{N}\alpha_i\,,\text{ where }\alpha_i\in V^{\otimes i}.$$

This is an (<u>associative</u>) algebra, with multiplication induced by tensoring elements together

$$\underbrace{(v_1\otimes\cdots\otimes v_s)}_{\substack{\rotatebox{90}{$\in$}\\ V^{\otimes s}}}\cdot\underbrace{(w_1\otimes\cdots\otimes w_t)}_{\substack{\rotatebox{90}{$\in$}\\ V^{\otimes t}}}:=\underbrace{v_1\otimes\cdots\otimes v_s\otimes w_1\otimes\cdots\otimes w_t}_{\substack{\rotatebox{90}{$\in$}\\ V^{\otimes s+t}}}.$$

extending by linearity to define $\bullet : V^{\otimes s}\times V^{\otimes t}\to V^{\otimes s+t}$.

and then $\bullet : T(V)\times T(V)\longrightarrow T(V)$.

This has an identity element $1\in\mathbb{R}$.

(by convention $V^{\otimes s}\times V^{\otimes 0}\longrightarrow V^{\otimes s}$
is the iso. $V^{\otimes s}\otimes\mathbb{R}\xrightarrow{\sim}V^{\otimes s}$
induced by scalar mult.).

e.g., if $V=\mathbb{R}^k$, then $T(V)\cong\mathbb{R}\langle x_1,\dots,x_k\rangle$ <u>non-commutative polynomials</u> in $x_1,\dots,x_k$ where $\{x_1,\dots,x_k\}\in(\mathbb{R}^k)^{\otimes 1}=\mathbb{R}^k$ standard basis.

(e.g., $x_1x_2x_3^2\neq x_3x_1x_3x_2$).

$V\rightsquigarrow T(V)$ is covariantly <u>functorial</u> in V (functor : $\mathrm{Vect}_{\mathbb{R}}\xrightarrow{T(-)}\mathrm{Alg}_{\mathbb{R}}$) :

a linear map $f: V \to W$ induces $f^{\otimes k}: V^{\otimes k} \to W^{\otimes k}$ and $\sum_{k\geq 0} f^{\otimes k}: T(V) \to T(W)$.

(compatible w/ composition, —)

exercise check f induces canonical linear map $f^{\otimes k}: V^{\otimes k} \to W^{\otimes k}$ which is an iso if f is, s.t., $(f\circ g)^{\otimes k} = f^{\otimes k} \circ g^{\otimes k}$.

In particular $\exists$ Lie group representation $GL(V) \xrightarrow{\rho} GL(V^{\otimes k})$ (exercise: it's smooth, hence a Lie group hom.)

$$f \longmapsto f^{\otimes k}$$

(can use this to associate to $\begin{matrix} E^{\text{rank } s} \\ \downarrow \\ M \end{matrix} \rightsquigarrow \begin{matrix} E^{\otimes k} \\ \downarrow \\ M \end{matrix}$ new vector bundle).. $\leftarrow E^{\otimes k} := E \times_\rho (\mathbb{R}^s)^{\otimes k}$.

Two interesting quotient algebras of $T(V)$.

Symmetric algebra: $Sym(V) := T(V)/I_{sym.}$ $\leftarrow$ ideal gen. by elts. of the form $v\otimes w - w \otimes v$.

$Sym^k(V)$ image of $V^{\otimes k}$ in $Sym(V) = T(V)/I$.

e.g., $Sym(\mathbb{R}^k) = \mathbb{R}[x_1, \dots, x_k]$. (commutative) polynomial ring.

Exterior algebra:

Define $\Lambda V := T(V)/I_{antisym.}$ $\leftarrow$ ideal gen. by elements of the form $v \otimes v$,

elements of $I_{antisym.}$ are finite sums of the form. $\alpha \cdot (v\otimes v) \cdot \beta$

α, β: elts. of $T(V)$,

Notation: $[\alpha_1 \otimes \alpha_2 \otimes \alpha_3]$ in $\Lambda(V)$ is denoted $\alpha_1 \wedge \alpha_2 \wedge \alpha_3$.

Note that since $v \wedge v = [v \otimes v] = 0$, $\Rightarrow$ (for any v,w) $(v+w)\wedge(v+w) = 0$

$$= \underbrace{\cancel{v\wedge v}}_{0} + w\wedge v + v \wedge w + \underbrace{\cancel{w \wedge w}}_{0}$$

$\Rightarrow$ for any v, w, $\boxed{v \wedge w = - w\wedge v}$ (✱)

If $\{v_1, \dots, v_k\}$ is a basis for V, elements of ΛV in image of $V^{\otimes s}$ can be expressed as $\left[\sum_{i_1, \dots, i_s} a_{i_1 \cdots i_k} v_{i_1} \otimes \cdots \otimes v_{i_s}\right]$

$$= \sum_{i_1, \dots, i_s} a_{i_1 \cdots i_k} v_{i_1} \wedge \cdots \wedge v_{i_s}.$$

$$\underset{\text{using } (\star)}{=} \sum_{i_1 < i_2 < \cdots < i_s} a'_{i_1 \cdots i_s} v_{i_1} \wedge \cdots \wedge v_{i_s}$$

↑ b/c if a given basis element is repeated in expression $v_{i_1} \wedge \cdots \wedge v_{i_s}$, that term is 0 b/c $v \wedge v = 0$.

e.g., $[5 v_1 \otimes v_2 + 2 v_1 \otimes v_1 + 10 v_2 \otimes v_1]$

$$= 5 v_1 \wedge v_2 + \overset{0}{\cancel{2 v_1 \wedge v_1}} + 10 v_2 \wedge v_1$$

$$= 5 v_1 \wedge v_2 - 10 v_1 \wedge v_2$$

$$= -5 v_1 \wedge v_2$$

Rmks: (1) $v_0 \wedge \cdots \wedge v \wedge \cdots \wedge v \wedge \cdots \wedge v_k = 0$

repeat.

(2) ΛV is an algebra:

e.g., $(v \wedge w) \cdot (x \wedge y) = v \wedge w \wedge x \wedge y$.

e.g., $\Lambda \mathbb{R}^k \cong \mathbb{R}[\varepsilon_1, \dots, \varepsilon_k] \Big/ \begin{matrix} \varepsilon_i^2 = 0 \\ \varepsilon_i \varepsilon_j = -\varepsilon_j \varepsilon_i . \end{matrix}$

where $\varepsilon_1, \dots, \varepsilon_k$ standard basis of $\mathbb{R}^k$.

Define: $\Lambda^s V :=$ degree s terms in in ΛV. (meaning image of $V^{\otimes s}$).

We can characterize $\Lambda^s V$ by a univ. property (a little different from one for $V^{\otimes s}$).

Def: A map $\psi : \underbrace{V \times \cdots \times V}_{s \text{ copies}} \longrightarrow Z$ is alternating multilinear if
(or antisymm. multilinear)

- ψ is multilinear. (linear in each slot, generalizing bilinear)
- $\psi(\text{---} v, \text{--}, v, \text{---}) = 0$ for all $v \in V$.

  any repeat entry

$\Rightarrow \quad \psi(\text{---}, v, \text{---}, w, \text{---}) = -\psi(\text{---}, w, \text{---}, v, \text{---})$

by plugging in $(\text{---}, v+w, \text{---}, v+w, \text{---})$ into

There is a canonical alt. multilinear map

$$\underbrace{V \times \cdots \times V}_{s} \xrightarrow{\psi} \Lambda^s V$$

$$(v_1, \dots, v_s) \longmapsto v_1 \wedge \cdots \wedge v_s.$$

and $(\Lambda^s V, \psi)$ satisfies the following universal property:

any alt. multilinear map

$$\begin{tikzcd}
\underbrace{V \times \cdots \times V}_{s} \arrow[r, "f"] \arrow[dr, "\psi"'] & Z \\
& \Lambda^s V \arrow[u, dashed, "\exists!\, \bar{f} \text{ linear map.}"']
\end{tikzcd}$$

factors uniquely through ψ via a linear map $\bar{f} : \Lambda^s V \to Z$ as in diagram.