Last time:

$\bigoplus_{k\geq 0} V^{\otimes k}$ $T(V)/I$ ← rel. gen. by $v\otimes v$.

Given V vec. space $/\mathbb{R}$ $\rightsquigarrow$ $V^{\otimes k}$, $T(V)$, $\Lambda(V)$, $\Lambda^s V$ (image of $V^{\otimes s} \to \Lambda(V)$).

Univ. property of wedge product:

There is a canonical alt. multilinear map

$$\underbrace{V\times\cdots\times V}_{s} \xrightarrow{\psi} \Lambda^s V$$

$$(v_1,\dots,v_s) \longmapsto v_1\wedge\cdots\wedge v_s. \quad (\text{image of } v_1\otimes\cdots\otimes v_s \text{ in } \Lambda^s V)$$

and $(\Lambda^s V, \psi)$ satisfies the following universal property:

any alt. multilinear map

$$\begin{tikzcd} \underbrace{V\times\cdots\times V}_{s} \arrow[r, "f"] \arrow[dr, "\psi"'] & Z \\ & \Lambda^s V \arrow[u, dashed, "\exists!\, \overline{f} \text{ linear map.}"'] \end{tikzcd}$$

factors uniquely through ψ via a linear map $\overline{f}: \Lambda^s V \to Z$ as in diagram.

Basis:

Prop: Given $\mathcal{B} = \{v_1,\dots,v_k\}$ basis for V, a basis $\Lambda^s V$ consists of

$$\Lambda^s\mathcal{B} = \{ v_{i_1}\wedge\cdots\wedge v_{i_s} \quad i_1<\cdots<i_s \}$$

In particular, $\dim \Lambda^s V = \begin{cases} 0 & \text{if } s>k=\dim(V) \\ \binom{\dim(V)=k}{s} & \text{if } s\leq k=\dim(V). \end{cases}$

ex: $\mathbb{R}^3$, basis $\{e_1, e_2, e_3\}$.

Then $\Lambda^0 \mathbb{R}^3 = \mathbb{R}$, basis 1. (1-dim'l)

$\Lambda^1 \mathbb{R}^3 = \mathbb{R}^3$, basis e_1, e_2, e_3. (3-dim'l).

$\Lambda^2 \mathbb{R}^3$ basis $e_1\wedge e_2, e_2\wedge e_3, e_1\wedge e_3$. ($\binom{3}{2} = 3$ dim'l)

$\Lambda^3 \mathbb{R}^3$ basis $e_1 \wedge e_2 \wedge e_3$. ($\binom{3}{3} = 1$ dim'l)

$\Lambda^4 \mathbb{R}^3 = \{0\}$.

($\Lambda^{\geq 4} \mathbb{R}^3 = \{0\}$)

Sketch proof of proposition:

- $\Lambda^s \mathcal{B}$ spans $\Lambda^s V$:

  Assume we knew $\{v_{i_1} \otimes \cdots \otimes v_{i_s} \mid i_1, \ldots, i_s \text{ arbitrary in } \{1, \ldots, k\}\}$ was a basis for $V^{\otimes s}$.

  $\Rightarrow$ representatives of in $\Lambda^s V := V^{\otimes s}/\sim$ span $\Lambda^s V$.

  But by applying $v \wedge v = 0$ and/or $v \wedge w = -w \wedge v$, we see that span of above collection $=$ span of $\Lambda^s \mathcal{B}$ in $\Lambda^s V$.

  $\Rightarrow \Lambda^s \mathcal{B}$ spans.

- Linear independence of $\Lambda^s \mathcal{B}$:

  - case $k = \dim(V) = s$, so $\Lambda^s \mathcal{B} = \{v_1 \wedge \cdots \wedge v_k\}$.

    We know $v_1 \wedge \cdots \wedge v_k$ spans, need to know its non-zero.

    Consider the alternating multilinear map

    $$\tau : \underbrace{V \times \cdots \times V}_{k} \longrightarrow \mathbb{R}$$

with $\tau(v_1,\dots,v_k)=1$; (claim: τ extends to a unique alternating & multilinear map; exercise.)

By univ. property, τ induces

$$\bar{\tau}: \Lambda^k V \longrightarrow \mathbb{R} \qquad \text{satisfying} \qquad \underbrace{\bar{\tau}\circ\psi(v_1,\dots,v_k)}_{=\bar{\tau}(v_1\wedge\cdots\wedge v_k)} = \underbrace{\tau(v_1,\dots,v_k)}_{=1}$$

$$v_1\wedge\cdots\wedge v_k \longmapsto 1.$$

In particular, $v_1\wedge\cdots\wedge v_k \neq 0$.

• if $s < k = \dim(V)$, we want to show

$\Lambda^s\mathcal{B} = \{v_{i_1}\wedge\cdots\wedge v_{i_s} \mid i_1<\cdots<i_s\}$ is linearly independent.

Suppose have a relation $\displaystyle\sum_{i_1<\cdots<i_s} a_{i_1\cdots i_s} v_{i_1}\wedge\cdots\wedge v_{i_s} = 0$. (✱✱✱)

Fix a particular $i_1<\cdots<i_s$, (want $a_{i_1\cdots i_s}=0$). Note $\exists$ a unique $j_1,\dots,j_{k-s}$ ($\{j_1,\dots,j_{k-s}\} = \{1,\dots,k\}\setminus\{i_1,\dots,i_s\}$)

such that (✱) $v_{i_1}\wedge\cdots\wedge v_{i_s} \wedge \underbrace{(v_{j_1}\wedge\cdots\wedge v_{j_{k-s}})} = \pm\, v_1\wedge\cdots\wedge v_k$.

(✱) (any other elt. of $\Lambda^s\mathcal{B}$) $\wedge (v_{j_1}\wedge\cdots\wedge v_{j_{k-s}}) = 0$.

Wedging the linear relation (✱✱✱) with $v_{j_1}\wedge\cdots\wedge v_{j_{k-s}}$, we learn

$$\pm\, a_{i_1\cdots i_s} v_1\wedge\cdots\wedge v_k = 0 \underset{(b/c\ v_1\wedge\cdots\wedge v_k\neq 0)}{\Longrightarrow} a_{i_1\cdots i_s}=0.$$

$i_1<\cdots<i_s$ arbitrary, so we're done.

$\boxtimes$.

A corollary of above argument is:

<u>Lem</u>: (wedge dependence lemma): $\forall\ v_1, \dots, v_s \in V$,

$$v_1, \dots, v_s \text{ is linearly independent} \iff v_1 \wedge \dots \wedge v_s \neq 0 \text{ in } \Lambda^s V.$$

<u>(exercise)</u>:

<u>Functoriality and determinant</u>:

Any map $T: V \to W$ induces $\Lambda^s T : \Lambda^s V \to \Lambda^s W$

(the corresp. alt. multilinear map $\underbrace{V \times \dots \times V}_{s} \longrightarrow \Lambda^s W$

sends $(v_1, \dots, v_s) \longmapsto Tv_1 \wedge \dots \wedge Tv_s$.

In particular on pure wedges, $\Lambda^s T(v_1 \wedge \dots \wedge v_s) = (Tv_1) \wedge \dots \wedge (Tv_s)$.

Now say $T: V \to V$ with $\dim(V) = k$.

1-dim'l vector spaces.

It follows that we get a map $\Lambda^k T : \Lambda^k V \longrightarrow \Lambda^k V$

$$\Lambda^k V = \Lambda^{top} V \qquad \Lambda^k V = \Lambda^{top} V.$$

However, a linear map from a 1-D vec. space Z to itself must be mult. by a scalar.

$\Rightarrow \Lambda^k T : \omega \longmapsto d\omega$ for some $d \in \mathbb{R}$.

<u>Def</u>: (<u>Determinant</u>).

$\det(T) :=$ the unique scalar $d \in \mathbb{R}$ such that $(\Lambda^k T)(\omega) = d\omega$ for any $\omega \in \Lambda^k V$ where $k = \dim(V)$.

<u>e.g.</u>, $T: \mathbb{R}^2 \to \mathbb{R}^2$
$e_1 \longmapsto a e_1 + c e_2$
$e_2 \longmapsto b e_1 + d e_2$

i.e., $T(\vec{v}) = A\vec{v}$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then $\Lambda^2 T(e_1 \wedge e_2) = T(e_1) \wedge T(e_2) = d\, e_1 \wedge e_2$ for some d. what's d?

<u>compute</u>: $T(e_1) \wedge (Te_2) = (ae_1 + ce_2) \wedge (be_1 + de_2)$

$$= ad\, e_1 \wedge e_2 + bc\, e_2 \wedge e_1$$
$$= (ad - bc)\, e_1 \wedge e_2.$$
$$\Rightarrow \det(T) = ad - bc.$$

Tensor calculus on manifolds

We've seen that from V we can functorially associate $V^{\otimes s}$, $\Lambda^s V$ for any s.

& moreover the associated maps

$$\begin{aligned} GL(V) &\xrightarrow{\rho^{\otimes s}} GL(V^{\otimes s}) \\ T &\longmapsto T^{\otimes s} \\ GL(V) &\xrightarrow{\rho^{\wedge s}} GL(\Lambda^s V) \\ T &\longmapsto \Lambda^s T. \end{aligned}$$

are Lie group representations (in particular, they're smooth).

We can therefore use $\rho^{\otimes s}$, $\rho^{\wedge s}$ to construct new vector bundles

$$\begin{array}{c} E^{\otimes s} \\ \downarrow \\ M \end{array}, \quad \begin{array}{c} \Lambda^s E \\ \downarrow \\ M \end{array} \quad \text{from an existing} \quad \begin{array}{c} E \\ \downarrow \pi \\ M \end{array} \quad \text{rank } k \text{ vector bundle.}$$

Here's a more direct construction of $\Lambda^s E$ ($E^{\otimes s}$ is similar).

As a set, $\Lambda^s E := \coprod_{p \in M} \Lambda^s(E_p) = \{(p, v) \mid p \in M, v \in \Lambda^s E_p\}$,

$$\begin{array}{ccc} \Lambda^s E & \ni & (p,v) \\ \pi \downarrow & & \downarrow \\ M & \ni & p. \end{array}$$

Choose a cover $\{U_\alpha\}$ of M over which there are

- charts on M $\phi_\alpha : U_\alpha \longrightarrow \phi_\alpha(U_\alpha) \overset{\text{open}}{\subseteq} \mathbb{R}^n$.
- trivializations of $E|_{U_\alpha}$, $\psi_\alpha : E|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{R}^k$.

$(p,v) \longmapsto (p, (\Psi_\alpha)(p)(v))$

We define a collection of "desired charts" of $\Lambda^s E$: Φ_α

$$\Lambda^s E|_{U_\alpha} \longrightarrow U_\alpha \times \Lambda^s \mathbb{R}^k \cong U_\alpha \times \mathbb{R}^{\binom{k}{s}} \xrightarrow{\phi_\alpha \times id} \phi_\alpha(U_\alpha) \times \mathbb{R}^{\binom{k}{s}}$$

$\uparrow$ id $\times$ (any linear iso.)

$$(p,v) \longmapsto (p, \Lambda^s(\underbrace{\Psi_\alpha(p)}) (v)).$$

$\Psi_\alpha(p) : E_p \longrightarrow \mathbb{R}^k$.

As we did for TM, T^*M, we'll use these "desired charts" to put a topology on $\Lambda^s E$ (by declaring each Φ_α to be a homeo., & each $\Lambda^s E|_{U_\alpha}$ to be open), and now need to check that

- Φ_α defines a smooth manifold str.,

  (transition functions associated to a pair of maps above, Φ_α, Φ_β?)

  $$\Phi_\beta \circ \Phi_\alpha^{-1} \Big|_{\phi_\alpha(U_\alpha \cap U_\beta) \times \Lambda^s \mathbb{R}^k} = \left(\phi_\beta \circ \phi_\alpha^{-1}, \ \Lambda^s(\psi_\beta \circ \phi_\beta^{-1} \circ \phi_\alpha \circ \psi_\alpha^{-1}) \right)$$

- π is smooth

- $\Lambda^s E$ is locally trivial, w/ linear fibers.

smooth b/c the map $GL(V) \to GL(\Lambda^s V)$, $T \longmapsto \Lambda^s T$ is smooth.