Today: focus primarily on $E = TM$, $E = T^*M$.

From last time, we can form $TM^{\otimes s}$, $(T^*M)^{\otimes s}$, $\Lambda^s T^*M$, $\Lambda^s TM$.

Let's focus on $\Lambda^k T^*M$, (dim $M = m$, so $\Lambda^0 T^*M = \underline{\mathbb{R}}$ ← trivial bundle.

study their sections

$\Lambda^1 T^*M = T^*M$

$\vdots$

$\Lambda^k T^*M$ has rank $\binom{m}{k}$

$\vdots$

$\Lambda^m T^*M$ is a line bundle.

$\Lambda^i T^*M = \underline{\{0\}}$ for $i > M$.)

Sections of $\Lambda^k T^*M$ are called (differential) k-forms (or differential forms of degree k).

(Sections of $\Lambda^k TM$ are called polyvectorfields).

When $M = \mathbb{R}^n$, we've previously seen that any 1-form is of the form $\alpha = \sum f_i \, dx_i$

Similarly, any k-form must be of the form

$$\omega = \sum_{i_1 < \cdots < i_k} g_{i_1 \cdots i_k} \, \underbrace{dx_{i_1} \wedge \cdots \wedge dx_{i_k}}_{} \qquad (*) \qquad \text{(exercise: verify)}$$

a smooth function.

the k-form whose value at $p \in \mathbb{R}^n$ is $\underbrace{(dx_{i_1})_p \wedge \cdots \wedge (dx_{i_k})_p}_{}$.

$\in T_p^* \mathbb{R}^m$.

$\in \Lambda^k T_p^* \mathbb{R}^m = (\Lambda^k T^* \mathbb{R}^m)_p$.

$dx_i : p \longmapsto (p, (dx_i)_p)$.
where $(dx_i)_p$ denotes
$[x_i - x_i(p)] \in \mathcal{F}_p / \mathcal{F}_p^2$,
$\mathcal{F}_p \subseteq C^\infty(p)$.
$(dx_i)_p \in \mathcal{F}_p/\mathcal{F}_p^2 = T_p^* \mathbb{R}^m$.

It follows that for any manifold M, any differential k-form $\tau \in \Gamma(M; \Lambda^k T^*M)$ must have the form $(*)$ in local coordinates with respect to some chart, Notation:

$$\Omega^k(M) := \Gamma(\Lambda^k T^*M),$$ the space of (differential) k-forms.

- a vector space over $\mathbb{R}$.
- moreover, a $C^\infty(M)$-module.

note this generalizes $\Omega^0(M) = \Gamma(\Lambda^0 T^*M = \underline{\mathbb{R}}) = C^\infty(M)$

$$\Omega^1(M) = \Gamma(\Lambda^1 T^*M = T^*M).$$

Pull back: Let $f: M \to N$ be a smooth map.

Then, there is a pull-back operation:

$$f^*: \Omega^k(N) \longrightarrow \Omega^k(M),$$

$$\omega \longmapsto f^*\omega : p \longmapsto (p, \overbrace{(f_p^*)^{\wedge k}(\underbrace{\omega_{f(p)}}_{\Lambda^k T^*_{f(p)} N})}^{\Lambda^k T^*_p M})$$

check (exercise): This sends $\Omega^k(N)$ to $\Omega^k(M)$,

i.e., the result is a $(\underline{C^\infty})$ section of $\Lambda^k T^*M$,

i.e., the result is smooth.

[recall: $(f_p)_* = df_p : T_pM \to T_{f(p)}N$.

$\rightsquigarrow (f_p)^* : T^*_{f(p)}N \to T^*_p M$

$\rightsquigarrow (f_p^*)^{\wedge k} : \Lambda^k T^*_{f(p)} N \to \Lambda^k T^*_p M$]

In local coordinates: say have

$$\tilde{f} : \mathbb{R}^m \longrightarrow \mathbb{R}^n \quad \text{smooth map}$$

(e.g., by pre/post-composing by charts from f above),

$$\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_n),$$

if $\omega \in \Omega^k(\mathbb{R}^n)$ was of the form $\omega = \sum_{i_1 < \dots < i_k} g_{i_1 \dots i_k} \, dx_{i_1} \wedge \dots \wedge dx_{i_k}$,

then $$f^*\omega = \sum \underbrace{(g_{i_1 \dots i_k} \circ f)}_{f^* g_{i_1 \dots i_k}} \underbrace{d\tilde{f}_{i_1}}_{d(x_{i_1} \circ \tilde{f}) = d(\tilde{f}^* x_{i_1})} \wedge \dots \wedge \underbrace{d\tilde{f}_{i_k}}_{d(x_{i_k} \circ \tilde{f} = \tilde{f}^* x_{i_k})}.$$

Exterior derivative

Recall we defined $d : \Omega^0(M) \to \Omega^1(M)$

$$\Omega^0(M) = C^\infty(M) = \Gamma(\Lambda^0 T^*M), \qquad \Omega^1(M) = \Gamma(T^*M)$$

We can define an extension:

$$d_{(k)} : \Omega^k(M) \to \Omega^{k+1}(M)$$

of $d = d_0 : \Omega^0(M) \to \Omega^1(M)$,

defined as follows:

<u>On Euclidean space:</u> $M = \mathbb{R}^m$ (or an open subset thereof)

(1) For $f \in \Omega^0(\mathbb{R}^m)$, recall df could be computed as:

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

(2) If $\omega = \sum f_I dx_I \in \Omega^k(\mathbb{R}^m)$, where $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ if $I = (i_1, \ldots, i_k)$.

define $d_k \omega = \sum_I df_I \wedge dx_I \in \Omega^{k+1}(\mathbb{R}^m)$.

(example: (1) $\omega = xyz\, dx + y\, dy$ on $\mathbb{R}^3$.

$$d\omega = d(xyz) \wedge dx + d(y) \wedge dy.$$

$$= \left(\cancel{\frac{\partial}{\partial x}(xyz) dx \wedge dx}^{\,0} + \frac{\partial}{\partial y}(xyz) dy \wedge dx + \frac{\partial}{\partial z}(xyz) dz \wedge dx \right) + \cancel{dy \wedge dy}^{\,0}$$

$$= -xz\, dx \wedge dy - xy\, dx \wedge dz.$$

(2) $\omega = f\, dx \wedge dy \wedge dz$ on $\mathbb{R}^3$.

$d\omega = df \wedge dx \wedge dy \wedge dz$ $(= 0$ b/c $\Lambda^4 T^*\mathbb{R}^3 = \{\underline{0}\}$, or...$)$

$= (f_x dx + f_y dy + f_z dz) \wedge dx \wedge dy \wedge dz = 0 + 0 + 0 = 0.$

To define d on a more general M:

Given $\omega \in \Omega^k(M)$, let's define $d\omega \in \Omega^{k+1}(M)$. Pick $p \in M$. Suffices to define $d\omega$ in some nhood of p (as p was arbitrary), provided we check well-definedness.

Choose a chart (U, ϕ) around p giving a diffeo.

$$\begin{tikzcd} U \arrow[r, shift left, "\phi", "\cong"'] & \phi(U) \arrow[l, shift left, "\phi^{-1}"] \underset{\text{open}}{\subseteq} \mathbb{R}^m, \end{tikzcd}$$

$$\begin{tikzcd} \Omega^k(\phi(U)) \arrow[rr, "d_k \text{ (defined above)}"] & & \Omega^{k+1}(\phi(U)) \arrow[d, "\phi^*"] \\ \Omega^k(U) \arrow[u, "(\phi^{-1})^*", "\cong"'] \arrow[rr, dashed, "d_k \text{ defined to make diagram commute.}"'] & & \Omega^{k+1}(U) \end{tikzcd}$$

$\omega|_U \in \Omega^k(U)$

Define $(d_k\omega)|_U := \phi^* d_k((\phi^{-1})^* \omega|_U)$

<u>Prop:</u> this is independent of chart chosen. (exercise), & gives a well-defined

$d_k : \Omega^k(M) \to \Omega^{k+1}(M).$

Frequently call d, suppress k.

<u>Example on $\mathbb{R}^3_{(x,y,z)}$:</u>

$$0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3 \longrightarrow 0$$

$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \quad$ ($\approx$ 'gradient')

$d(f dx + g dy + h dz) = \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right) dx \wedge dz \quad$ ($\approx$ 'curl')

$+ \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$

$+ \left(\frac{\partial h}{\partial x} - \frac{\partial g}{\ldots}\right) dy \wedge dz.$

$$d(f_1\, dy \wedge dz + f_2\, dz \wedge dx + f_3\, dx \wedge dy)$$

$$= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) dx \wedge dy \wedge dz.$$ ($\approx$ 'divergence').

OH: (Zoom) M 12:45-1:45pm, W 2-3, Th 4:30-5:30.
