Last time: $M^m$ manifold $\rightsquigarrow \Lambda^k T^*M \rightsquigarrow \Omega^k(M) := \Gamma(\Lambda^k T^*M)$.

space of (differential) k-forms

$\exists$ a map $d_{(k)} : \Omega^k(M) \to \Omega^{k+1}(M)$. exterior derivative

Some basic properties of exterior differentiation:

There's an operation $\wedge : \Omega^k(M) \times \Omega^l(M) \longrightarrow \Omega^{k+l}(M)$

$$\omega \quad \sigma \longmapsto \omega \wedge \sigma : p \mapsto (p, \omega_p \wedge \sigma_p).$$

[ Rmk: comes from $\wedge : \Lambda^k V \times \Lambda^l V \to \Lambda^{k+l} V$

$$\alpha \quad \beta \longmapsto \alpha \wedge \beta.$$

This operation extends to the case $k=0$ and/or $l=0$, with the convention that for $\alpha \in \Lambda^0 V = \mathbb{R}$, $\beta \in \Lambda^l V$, $\alpha \wedge \beta := \alpha\beta$. ]

Similarly $\wedge : \Omega^0(M) \times \Omega^l(M) \longrightarrow \Omega^l(M)$

$$f, \quad \alpha \longmapsto f \wedge \alpha := f\alpha.$$

(and similarly if $l=0$ instead of $k$).

Lemma: (extends fact that $d(fg) = f\,dg + g\,df$).

$d$ satisfies the (skew-commutative) Leibniz rule:

"graded"?

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta) \quad \text{where } \alpha \in \Omega^k(M),\ \beta \in \Omega^l(M)$$

Lemma: $d$ commutes with pullbacks: if $f: M \to N$ smooth map, then this diagram commutes:

$$\begin{tikzcd}
\Omega^k(N) \arrow[r, "f^*"] \arrow[d, "d_k"'] \arrow[dr, phantom, "\circlearrowleft"] & \Omega^k(M) \arrow[d, "d_k"] \\
\Omega^{k+1}(N) \arrow[r, "f^*"] & \Omega^{k+1}(M)
\end{tikzcd}$$

$\mathrm{im}(d_{k-1}) \subseteq \ker(d_k)$, but might not be equal!

Lemma: $d_k \circ d_{k-1} = 0$ for any $k$.

$M^m$: $\underset{0}{\Omega^{-1}(M)} \xrightarrow{d_1 \equiv 0} \Omega^0(M) = C^\infty(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \cdots\cdots \longrightarrow \Omega^m(M) \xrightarrow{d_m \equiv 0} \underset{0}{\Omega^{m+1}(M)} \to 0$.

Proof: (sketch). First case is $k=1$: $\Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M)$.

To check that $d_1 \circ d_0(f) \equiv 0$, we need to check $\equiv 0$ near any point

$\rightsquigarrow$ reduce to showing $d_1 \circ d_0 \equiv 0$ on $M = \mathbb{R}^m$.

On $\mathbb{R}^m$: Let $f \in \Omega^0(M)$.

$$\rightsquigarrow df = \sum_{i=1}^m \frac{\partial f}{\partial x_i} dx_i .$$

$$\rightsquigarrow d(df) = \sum_{i=1}^m d\left(\frac{\partial f}{\partial x_i}\right) \wedge dx_i .$$

$$= \sum_{i=1}^m \left( \sum_{j=1}^m \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j \right) \wedge dx_i .$$

$$= \sum_{i=1}^m \sum_{j=1}^m \underbrace{\frac{\partial^2 f}{\partial x_i \partial x_j}} \underbrace{dx_j \wedge dx_i}_{\equiv 0 \text{ if } i=j} .$$

$$= \sum_{j<i} \left( \frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i} \right) dx_j \wedge dx_i$$

$= 0$ (by Clairaut's thm / equality of mixed partial derivatives)

For general $k$-forms, again reduce to $M = \mathbb{R}^m_{(x_1,\dots,x_m)}$, have $dx_1, \dots, dx_m$.

- if $\alpha = dx_I$ (means $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$), note $d\alpha = 0$.
- if $\alpha = f_I \, dx_I$, then $d(\alpha) = df_I \wedge dx_I$

$$\& \quad d(d\alpha) \underset{\text{Leibniz}}{=} \underbrace{d(df_I)} \wedge dx_I + (-1)^{\text{sign}} df_I \wedge \underbrace{d(dx_I)}_{0 \text{ by above}} .$$

we defined
$d\left(\sum_I f_I dx_I\right) := \sum_I df_I \wedge dx_I$.
so $d(dx_I = 1\, dx_I) = d(1) \wedge dx_I = 0$.

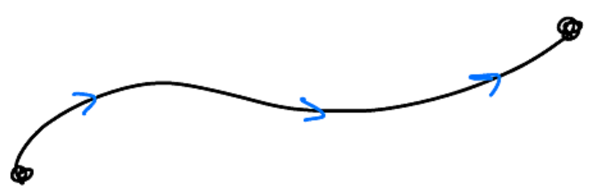
$= 0$ b/c $d(d(\text{function})) = 0$ by previous case.

- $d^2$ is linear $\Rightarrow$ since a general $\alpha \in \Omega^k(\mathbb{R}^n)$ is of the form $\alpha = \sum_I f_I dx_I$,
  $\Rightarrow d^2(\alpha) = 0$.

We'll return to studying applications of exterior differentiation shortly. First, let's talk about <u>integration of 1-forms</u>, and <u>orientations</u>.

<u>Integrating 1-forms</u>:

Let $C = \mathrm{im}(\gamma : I \xrightarrow{\text{embedding}} M)$. Assume $C$ is <u>oriented</u>, meaning we have chosen,
closed interval
at each $p \in C$ a direction in $T_pC$ tangent to $C$, smoothly varying in $p$.



$C$

(get an orientation on $C$ from a vector field, & can get a vector field via $\gamma : I \xrightarrow{\cong} C$ by taking $\gamma_*\left(\frac{\partial}{\partial t}\right)$, but two vector fields differing by mult. by a <u>positive</u> function induce same orientation on $C$).

Let $\underline{\omega \in \Omega^1(M)}$. Then we can define

$$\int_C \omega := \int_c^d \underbrace{\gamma^* \omega}_{\in \Omega^1(I)} \quad \text{where } \gamma : I = [c,d] \longrightarrow M$$

with image $C$, inducing the orientation.

(note: given a 1-form on $I = [c,d]$, $f(t)dt = \alpha$
can define $\int_c^d \alpha$ using standard calculus).

**Lemma:** This definition doesn't depend on the particular choice of orientation-preserving parametrization $\gamma : [c,d] \to M$ of $C$.

**Pf:** Take a different $\gamma_1 : [a,b] \to M$ parametrizing $C$.

Then, $\exists$ an orientation preserving diffeo.

$$g : [a,b] \xrightarrow{\cong} [c,d] \quad \text{such that} \quad \gamma_1 = \gamma \circ g.$$

(so $g(a) = c$, $g(b) = d$ )

Now $\gamma_1^* \omega = (\gamma \circ g)^* \omega = g^* \gamma^* \omega$.

so:
$$\int_c^d \gamma^* \omega = \int_c^d f(t)\,dt \overset{t = g(s)}{=} \int_a^b f(g(s))\, dg(s) = \int_a^b g^*(f(t)\,dt)$$

(where $\gamma^* \omega = f(t)\,dt$.)

$$= \int_a^b g^* \gamma^* \omega$$

$$= \int_a^b \gamma_1^* \omega.$$

$\blacksquare$

Soon, we will show how to integrate $k$-forms on $k$-dim'l oriented submanifolds.

Next time: orientations of manifolds (without requiring dimension $= 1$.)