Orientations and orientability

Linear algebra: V vector space of dimension n.

An orientation of V is

- a choice of equivalence class of basis $(v_1, \dots, v_n)$ under the equiv. relation
  $(v_1, \dots, v_n) \sim (w_1, \dots, w_n)$ if the linear map $T: V \to V$, $v_i \mapsto w_i$ has $\det(T) > 0$.

$\Updownarrow$ (exercise: check these are equivalent). when $\dim V > 0$!

- a choice of generator $\alpha \in \Lambda^n V$ (in particular non-zero), up to the equiv. relation of real positive scaling.

The set of orientations of V is $or(V) := \left(\Lambda^{n} V \setminus \{0\}\right) \big/ \mathbb{R}_+ = \left\{ \text{basis of } V \right\} \big/ \sim$ ($n = \dim(V)$)

{equivalently, a choice of connected component of $\Lambda^n V \setminus 0$}

Note: when $\dim V = 0$, so $V = \{0\}$, $\Lambda^{\dim V} V = \mathbb{R}$, so the notion of an orientation (second version) still makes sense, and is simply equivalent to a choice of sign $\{+, -\}$.

i.e., $or(\{0\}) = \{+, -\}$. ($\cong \mathbb{Z}/2$ as a $\mathbb{Z}/2$-set).

For any vector space V, $or(V)$ has two elements, but cannot be canonically identified w/ $\{+, -\}$.

It is however a free $\mathbb{Z}/2$-set of cardinality 2. under the map $\underset{\substack{\text{gen. of}\\ \mathbb{Z}/2}}{\tau} \cdot [\alpha] := [-\alpha]$.

e.g., $or(\mathbb{R}^2) = \{[e_1 \wedge e_2], [e_2 \wedge e_1]\}$.

"$\mathbb{Z}/2$-torsor".

A pair $(V, \underset{or(V)}{\sigma})$ is called an oriented vector space

A linear iso, $T: (V, \sigma_V) \xrightarrow{\cong} (W, \sigma_W)$ is orientation-preserving if $\underbrace{T_* \sigma_V}_{= [\Lambda^{\dim V} T(\alpha)]} = \sigma_W$. ($\sigma_V = [\alpha]$)

Observe: $T: (V, \sigma_V) \to (V, \sigma_V)$ is orientation preserving iff $\det(T) > 0$. iff $T: (V, -\sigma_V) \to (V, -\sigma_V)$ is orientation preserving.

Therefore we simply say $T: V \to V$ is orientation preserving if $\det(T) > 0$.

Now, let M^m be a connected manifold, $E \to M^m$ a vector bundle of rank n.

We say $E \to M$ is <u>orientable</u> if $\Lambda^{\mathrm{rank}(E)=n} E \setminus$ (zero section) has exactly two components.

image of $p \mapsto (p,0)$, $M \to \Lambda^n E$.

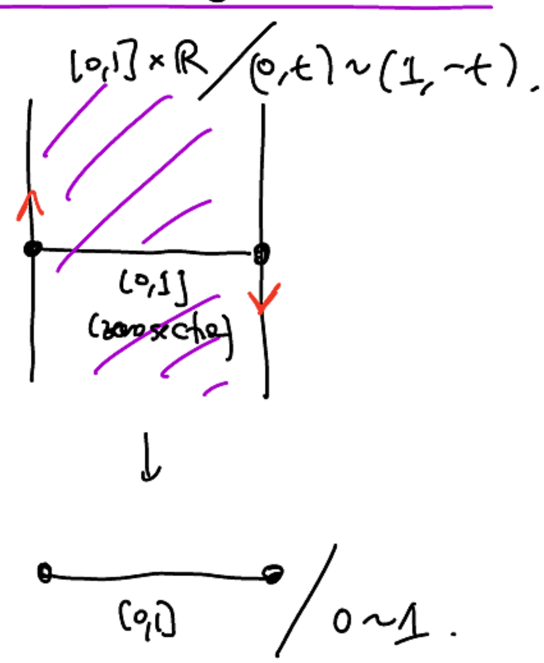
(know it has at most two components, essentially b/c $\Lambda^n E_p \setminus \{0\}$ has two components for every p. — <u>exercise</u>.)

If $E \to M$ is orientable, an <u>orientation</u> of E is a choice of component of $\Lambda^{top} E \setminus$ zero section.

<u>Rmk</u>: $E \to M$ is orientable iff there exists a nowhere vanishing $s \in \Gamma(\Lambda^{top} E)$ ($\Leftrightarrow$ $\Lambda^{top} E$ is trivializable.)

(exercise, $\Leftarrow$ is more straightforward)

<u>Non-ex</u>: E = Möbius-bundle (rank 1)
$\downarrow$
S^1.



<u>Def</u>: Say M is <u>orientable</u> if $TM \to M$ is orientable. <u>An orientation of M</u> is by definition an orientation of TM.

(<u>Rmk</u>: Linear algebra lemma states that given $0 \to V \xrightarrow{i} W \xrightarrow{j} Z \to 0$ "exact sequence" meaning j sends $\mathrm{im}(i)$ to 0 and induces $W/V \xrightarrow[{[j]}]{\cong} Z$. ($\mathrm{im}(i) = \ker(j)$),

e.g., $0 \to V \underset{\text{subspace}}{\hookrightarrow} W \twoheadrightarrow W/V \to 0$

then an orientation of 2/3 of vector spaces canonically determines an orientation of third one)

$\Rightarrow$ $N \subseteq M$ submanifold, then an orientation of M & an orientation of

$\nu N := TM|_N / TN$. determine an orientation of N (in particular orientability of M & νN $\Rightarrow$ orientability of N)),

normal bundle

Prop: TFAE:

(a) $E \to M$ orientable

(b) $\exists$ a trivializing cover $\{U_\alpha\}$ of M along w/ trivializations $\Phi_\alpha : E|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times \mathbb{R}^n$ s.t. the transition functions $\Phi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(n, \mathbb{R})$ factors through $GL^+(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) \mid \det(A) > 0\}$.

(c) There is a nowhere vanishing section $s \in \Gamma(\Lambda^{top} E)$.

Rmk: Moreover, the prop + ε will show that if E orientable in any sense above the

$or(E) = \{$connected components of $\Lambda^{top}E \setminus \{$image of 0 section$\}\} = \{$non-zero sections $s \in \Gamma(\Lambda^{top}E)$ up to $\sim$, where $s \sim s'$ if $s = fs'$ where $f > 0$ everywhere$\}$.

orientations of E

$= \{$choices of trivializing cover & trivializations w/ transition functions in $GL^+\} / \sim$

same sort of $\sim$ appearing in classification of vector bundles by gluing data.

Sketch of proof:

WLOG, assume M is connected.

(a) $\Rightarrow$ (b). Say E orientable, & choose an orientation, i.e., a component Λ of $\Lambda^{top}E \setminus im(\underline{0})$.

Note for every $p \in M$, $\Lambda \cap (\Lambda^{top} E_{top})_p$ is precisely one of the two components of $\Lambda^{top}(E_p) \setminus \{0\}$; i.e., gives an element of $or(E_p)$.

Now, pick a cover $\{U_\alpha\}$ over which E is trivial, & consider only trivializations

$\Phi_\alpha : E|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{R}^n =: \underline{\mathbb{R}}^n$ over U_α

$$\begin{array}{ccc} E|_{U_\alpha} & \xrightarrow{\Phi_\alpha} & \underline{\mathbb{R}}^n|_{U_\alpha} \\ s \uparrow & & \\ U_\alpha & \dashrightarrow & (\Phi_\alpha)_* s := \Phi_\alpha \circ s \end{array}$$

which send $\Lambda \cap \Lambda^{top}(E|_{U_\alpha})$ to $[e_1 \wedge \cdots \wedge e_n]$

chosen orientation of E.

orientation of $\mathbb{R}^n$ induced by component containing constant section: $p \longmapsto (p, e_1 \wedge \cdots \wedge e_n)$.

Then, check each $\Phi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$ lands in $GL^+(n, \mathbb{R})$

(as it sends $[e_1 \wedge \cdots \wedge e_n] \to [e_1 \wedge \cdots \wedge e_n]$ in $(\mathbb{R}^n)_p$, i.e., orientation preserving $\forall p \in U_\alpha$).

(b) $\Rightarrow$ (c). Given $\{U_\alpha\}$, Φ_α "positive determinant" trivialization of E as in (b)

- choose a partition of unity $\{\varphi_\alpha\}$ subordinate to U_α.

- have $\overline{\Phi}_\alpha : E|_{U_\alpha} \xrightarrow{\cong} \underline{\mathbb{R}^n}|_{U_\alpha}$, & on RHS

have $\omega_\alpha = [\underline{e_1 \wedge \cdots \wedge e_n}]$ const. section of $\Lambda^n(\underline{\mathbb{R}}^n|_{U_\alpha})$.

$\rightsquigarrow (\Phi_\alpha^{-1})_* \omega_\alpha \in \Gamma(\Lambda^n E|_{U_\alpha})$

consider $\sum \varphi_\alpha (\Phi_\alpha^{-1})_* \omega_\alpha$; thought of as a section of $\Lambda^n E$ over M.

(extend $\varphi_\alpha (\Phi_\alpha^{-1})_* \omega_\alpha$ to be 0-section outside U_α.

<u>Claim</u>:

- ω vanishes nowhere; At each $p \in M$,

ω_p is a <u>finite</u> sum of $\varphi_\alpha(p) (\Phi_\alpha^{-1})_* (\omega_\alpha)_p$. $\omega_p \neq 0$?

Picking a particular $_{p \in} U_\beta, \Phi_\beta$ to push forward along

$\omega_p \neq 0 \iff (\Phi_{\beta *} \omega)_p \neq 0 \iff$

$$\sum_{\substack{\alpha \neq \beta \\ \text{finite} \\ \text{(b/c finitely} \\ \text{many } \varphi_\alpha(p) \neq 0)}} \underset{\text{positive.}}{\varphi_\alpha(p)} \left(\Phi_\beta \Phi_\alpha^{-1}\right)_* \left(e_1 \wedge \cdots \wedge e_n\right)_p + \underset{\text{positive}}{\varphi_\beta(p)} \left(e_1 \wedge \cdots \wedge e_n\right)_p.$$

Obs: $\left(\Phi_\beta \Phi_\alpha^{-1}\right)_* e_1 \wedge \cdots \wedge e_n$ = a pos. multiple of $e_1 \wedge \cdots \wedge e_n$.
(depending on p)

b/c $\det\left(\Phi_\beta \Phi_\alpha^{-1}\right)(p) > 0$.

$\Rightarrow \omega_p \neq 0$ as desired. ✓

(c) $\Rightarrow$ (a). Say $\omega \in \Gamma(\Lambda^{top} E)$ nowhere vanishing, so $\omega_p \neq 0$ $\forall p$.

Let $\Lambda^+ = \{(p, \alpha_p) \in \Lambda^{top} E \mid \alpha_p = c\,\omega_p \text{ for some } c > 0\}$.

$\Lambda^+ \subset \Lambda^{top} E$, $\alpha_p \in (\Lambda^{top} E)_p$.

$\Lambda^- = \{(p, \alpha_p) \in \Lambda^{top} E \mid \alpha_p = c\,\omega_p \text{ for some } c < 0\}$.

$\alpha_p \in (\Lambda^{top} E)_p$.

Then, note: $\Lambda^{top} E \setminus \{\underset{\text{image of}}{0} \text{ section}\} = \Lambda^+ \amalg \Lambda^-$, so $\Lambda^{top} E \setminus \{0 \text{ section}\}$ is disconnected $\Rightarrow E \to M$ orientable.

$\boxtimes$.