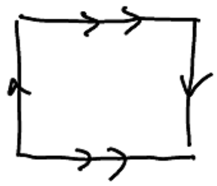
Examples: (non-orientable & orientable bundles/manifolds)

(1) (previously): $\begin{array}{c} E = \text{Mobius bundle} \\ \downarrow \\ S^1 \end{array}$ is not an orientable vector bundle.

In fact, if $M = E$ is not orientable as a manifold (meaning $TM \to M$ is not orientable).

(2) Klein bottle $K := [0,1]_s \times [0,1]_t \Big/ \begin{array}{l} (s,0) \sim (s,1) \\ (0,t) \sim (1, 1-t) \end{array}$



K is not orientable (exercise).

(3) $\mathbb{R}P^2 = \mathbb{R}^3 \setminus 0 \big/ \sim$ ← where $x \sim tx$ for all $t \in \mathbb{R}\setminus 0$.

is not orientable.

Note $S^2 \subset \mathbb{R}^3 \setminus 0$ unit sphere, & $\mathbb{R}P^2$ can be thought of as $S^2 \big/ x \sim -x$.

By following an oriented basis from x to $-x$, we find that $\mathbb{R}P^2$ is not orientable. (one method).

Observe S^2 is oriented & $\exists$ a $2:1$ map $S^2 \to \mathbb{R}P^2$, called the orientation double cover of $\mathbb{R}P^2$.

Classification of compact 2-manifolds (surfaces): up to diffeomorphism, if M^2 cpct connected manifold,

then $M \underset{\text{diffeo.}}{\cong}$ to one of the following:

if M orientable:

$\underset{\Sigma_0}{\overset{\shortparallel}{S^2}}, \; \underset{\Sigma_1}{\overset{\shortparallel}{T^2}}, \; \Sigma_g$ any $g \geq 2$. surface of genus g.

if M not orientable:

$\mathbb{RP}^2$, K, and one whose orientation double cover is Σ_g.

<u>De Rham cohomology</u> Previously, constructed:

M^m smooth manifold

$\rightsquigarrow T^*M$

$\rightsquigarrow \Lambda^k T^*M$

$\rightsquigarrow \Omega^k(M) := \Gamma(\Lambda^k T^*M)$ space of <u>differential k-forms.</u>

$(\text{w/ } \Omega^{-i} = \{0\}, \; \Omega^i = \{0\} \text{ for } i > \dim(M), \; \Omega^0 = C^\infty(M))$.

and

$d_{(k)} : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ <u>exterior derivative</u>

(shorthand $\Omega^i := \Omega^i(M)$), satisfying:

- if $\alpha \in \Omega^k$, $\beta \in \Omega^\ell$, $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + \alpha \wedge (-1)^k d\beta$.
- $d \circ f^* = f^* \circ d$ for any smooth $f : M \to N$.
- $d \circ d$ (means $d_k \circ d_{k-1}$) $= 0$.

Let's consider the sequence of vector spaces

$$(*) \qquad 0 \longrightarrow \Omega^0 \xrightarrow{d_0} \Omega^1 \xrightarrow{d_1} \cdots \cdots \longrightarrow \Omega^m \xrightarrow{d_m} 0$$

Ω^{k-1} Ω^{k+1}

Since $d_k \circ d_{k-1} = 0$ this means

$$\mathrm{Im}(d_{k-1}) \subset \ker(d_k).$$

Def: the kth de Rham cohomology group of M is given by

$$H^k(M) := H^k_{dR}(M) := \frac{\ker(d_k)}{\mathrm{im}(d_{k-1})}.$$

(where $\mathrm{im}(d_{k-1}) \subset \ker(d_k) \subset \Omega^k$)

Def: $\omega \in \Omega^k$ is closed if $d\omega = 0$ (so if $\omega \in \ker(d_k)$).

" " exact if $\omega = d\alpha$ (so $\omega \in \mathrm{im}(d_{k-1})$).

& note $H^k_{dR}(M) := \dfrac{\text{closed } k\text{-forms}}{\text{exact } k\text{-forms}}.$

Facts: (some of which we'll prove).

- de Rham cohomology groups are (contravariantly) functorial in M, and in particular diffeomorphism invariants

  (i.e., $f: M \to N$ induces $f^*: H^k_{dR}(N) \to H^k(M)$ compat. w/ composition, & $\mathrm{id}^* = \mathrm{id}$. $\Rightarrow$ if $f: M \xrightarrow{\cong} N$, then $f^*: H^k_{dR}(N) \xrightarrow{\cong} H^k_{dR}(M)$)

  (cor: if $H^k(M) \neq H^k(N)$ i.e., $\nexists$ iso, e.g., if dimensions different, then $\nexists\, M \xrightarrow{\cong} N$).

- de Rham coh. is finite dimensional $\forall$ degree k if M compact.