Last time: M^n smooth manifold, we defined its k^{th} <u>de Rham cohomology</u>, $k \geq 0$, as

$$H^k_{dR}(M) := \frac{\ker d_k}{\operatorname{im} d_{k-1}} = \frac{\text{closed } k\text{-forms}}{\text{exact } k\text{-forms}}$$

where $d_k : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$

$d_{k-1} : \Omega^{k-1}(M) \longrightarrow \Omega^k(M)$

& $d_k \circ d_{k-1} = 0 \Rightarrow \underbrace{\operatorname{im} d_{k-1}}_{\text{"exact } k\text{-forms"}} \subseteq \underbrace{\ker d_k}_{\text{"closed } k\text{-forms"}} \subseteq \Omega^k(M)$

<u>Examples</u>/<u>initial computations of de Rham cohomology</u>:

shorthand $\Omega^i := \Omega^i(M)$

(1) $M = \{pt.\}$ (0-dim'l manifold.)

$\Omega^0 = C^\infty(\{pt\}) \cong \mathbb{R}$,

Ω^i for $i \neq 0 \equiv 0$. ($\Rightarrow \operatorname{im} d_{i-1} = \ker d_i = 0$ for $i \neq 0$ $\Rightarrow H^i_{dR}(M) = 0$ for $i \neq 0$)

so $0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots$ becomes

$$0 \xrightarrow{d_{-1} \equiv 0} \mathbb{R} \xrightarrow{d \equiv 0} 0 \longrightarrow \cdots$$

Note: $\ker d_0 = \mathbb{R}$
$\cup$
$\operatorname{im} d_{-1} = 0$

$\Rightarrow H^0_{dR}(M) = \mathbb{R}$.

So $H^i_{dR}(M) = \begin{cases} \mathbb{R} & i = 0 \\ 0 & \text{otherwise.} \end{cases}$

2) $M = \mathbb{R}$.

$\Omega^0(M) = C^\infty(\mathbb{R})$

$\Omega^1(M) \xleftarrow{\cong} C^\infty(\mathbb{R})$ (every 1-form on M is $f\,dx$ some $f : \mathbb{R} \to \mathbb{R}$.)

$f\,dx \longleftarrow f$

& $d : \Omega^0 \longrightarrow \Omega^1$,

$$f \longmapsto \frac{df}{dx}dx \ (= df)$$

under the identification $\Omega^1 \cong C^\infty(\mathbb{R})$,

becomes

$$C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$$
$$f \longmapsto f'.$$

So the "de Rham complex" meaning $0 \xrightarrow{d} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots$

becomes in this case:

$$\cdots \to 0 \xrightarrow{d_{-1}\equiv 0} C^\infty(\mathbb{R}) \xrightarrow{d_0 : f \longmapsto f'} C^\infty(\mathbb{R}) \xrightarrow{d_1 \equiv 0} 0 \to 0 \to \cdots$$

deg -1 deg 0 deg 1

Note: $\ker(d_0) = \{\text{constant functions}\} \cong \mathbb{R}$.

$$\Rightarrow H^0(\mathbb{R}) = \frac{\ker(d_0)}{\cancel{\operatorname{im}(d_{-1})}\ 0} = \mathbb{R}/0 = \mathbb{R}.$$

$$H^1(\mathbb{R}) = \frac{\ker(d_1)}{\operatorname{im}(d_0)} = \frac{C^\infty(\mathbb{R})}{\operatorname{im}(d_0)} \underset{\text{claim}}{=} 0.$$

(why? Note that every $g \in C^\infty(\mathbb{R})$ is equal to f' for some f,
namely take $f(x) = \int_a^x g(u)\,du$, by FTC).
i.e., $\operatorname{im}(d_0) = C^\infty(\mathbb{R}) \cong \Omega^1$).

$H^i(\mathbb{R}) = 0$ for $i \neq 0,1$ b/c $\Omega^i(\mathbb{R}) = 0$ for $i \neq 0,1$.

$$\Rightarrow H^i(\mathbb{R}) = \begin{cases} \mathbb{R} & i = 0 \\ 0 & \text{otherwise} \end{cases},$$

3) $M = S^1$.

View S^1 as $\mathbb{R}/\mathbb{Z}$, w/ projection map $\mathbb{R} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$.

π induces $\pi^* : C^\infty(S^1) \longrightarrow C^\infty(\mathbb{R})$,

$$f \longmapsto f \circ \pi.$$

which is an injective map, inducing an identification

$$C^\infty(S^1) = \Omega^0(S^1) \xrightarrow[\pi^*]{\cong} \{\text{periodic functions on } \mathbb{R} \text{ with period } 1\}$$

$\swarrow d_0$ $\qquad\qquad\qquad \swarrow d_0$

Similarly, $\Omega^1(S^1) \xrightarrow[\pi^*]{\sim} \{f\,dx, \text{ where } f \text{ is periodic with period } 1\} \xleftarrow{\cong} \{\text{period fns. w/ period } 1\}$

$f\,dx \longleftarrow f$

(observe $\Omega^i(S^1) \equiv 0$ for $i \neq 0, 1$

$\Rightarrow H^i(S^1) = 0$ for $i \neq 0, 1$).

note: all constant fns. are periodic so $\ker(d_0|_{\text{per. fns.}}) = (\ker d_0)$

Using above identification:

- $H^0(S^1) = \dfrac{\ker d_0}{\operatorname{im} d_{-1} \equiv 0} = \dfrac{\text{constant functions}}{0} \cong \mathbb{R}.$

Now - $H^1(S^1) = \dfrac{\ker d_1}{\operatorname{im} d_0} \cong \dfrac{\{\text{periodic fns w/ period } 1\}}{\{\text{those period fns. } g \text{ which equal } f' \text{ where } f \text{ is periodic too}\}},$

i.e., $\operatorname{im} d_0 \cong \{C^\infty \text{ fxns. } g(x) \text{ which are } 1\text{-periodic and satisfy}$

$$\int_0^1 g(x)\,dx = 0\}.$$

$\overset{\text{exercise}}{\Longrightarrow}$ this implies that $H^1(S^1) \cong \mathbb{R}$,

(can show this by showing $\exists$ <u>exact sequence</u>

$$0 \to \mathbb{R} \to \{1\text{-period. fxns}\} \xrightarrow{\frac{d}{dx}} \{1\text{-periodic fxns}\} \xrightarrow{\int_0^1} \mathbb{R} \to 0.$$

in particular $\int_0^1 (-)\,dx$ sends $\operatorname{im}\left(\frac{d}{dx}\Big|_{1\text{-period. fxns}}\right) \to 0$

& induces an iso: $H^1(S^1) = \dfrac{1\text{-period fxns}}{\operatorname{im}\left(\frac{d}{dx}\Big|_{1\text{-period. fxns}}\right)} \xrightarrow[\int_0^1]{\cong} \mathbb{R}.$).

surjectivity ✓.

need to show injectivity)

Lemma: $H^0(M) \cong \mathbb{R}$ if M is connected.

Pf: For $f \in C^\infty(M) = \Omega^0(M)$, $df = 0 \iff f$ is locally constant $\underset{(M \text{ connected})}{\iff}$ f is constant.

$\Rightarrow H^0(M) = \dfrac{\ker d_0}{\cancel{\operatorname{im} d_{-1}}\ 0} = \{\text{constant fcns}\} \cong \mathbb{R}$. $\boxed{}$

Lemma: If $M = M_1 \sqcup M_2$, then $H^k(M) = H^k(M_1) \oplus H^k(M_2)$, for every $k \geq 0$.

Pf: exercise.

Functoriality: (contravariant)

Let $f: M \to N$ any smooth map.

Prop: There is a linear map $f^*: H^k(N) \to H^k(M)$ for all k.

Moreover
- if $f: M \xrightarrow{id} M$ then $f^* = id_{H^k(M)}$.
- if $f: M \to N$, $g: N \to P$, then
$$(g \circ f)^* = f^* \circ g^* : H^k(P) \to H^k(M).$$

Sketch:

Note f induces $f^*: \Omega^k(N) \to \Omega^k(M)$ satisfying $f^* \circ \underset{\text{on } N}{d} = \underset{\text{on } M}{d} \circ f^*$

To construct $f^*: \underset{\dfrac{\ker d_k}{\operatorname{im} d_{k-1}}}{\overset{\alpha}{H^k(N)}} \to H^k(M)$:

observe if $\alpha = [\omega]$ where $\omega \in \Omega^k(N)$ is closed, so $d\omega = 0$,
& $\omega \sim \omega'$ if $\omega - \omega' = d\tau$.

pick a representative $\omega \in \alpha$ which is closed by def'n, so $d\omega = 0$.

Apply f^* to get $f^*\omega \in \Omega^k(M)$, & note $d(f^*\omega) = f^*(d\omega) = f^*(0) = 0$, so $f^*\omega$ remains closed.

(i.e., f^* induces $f^*: \{\text{closed } k\text{-forms on } N\} \longrightarrow \{\text{closed } k\text{-forms on } M\}$.)

Now, if a closed k-form σ was exact, $\sigma = d\tau$, then note $f^*\sigma = f^*d\tau = d(f^*\tau)$, so $f^*\sigma$ is exact too.

so f^* induces

$$\begin{array}{ccc} f^*: \{\text{closed } k\text{-forms on } N\} & \longrightarrow & \{\text{closed } k\text{-forms on } M\}. \\ \cup\!| & \circlearrowright & \cup\!| \\ \{\text{exact } k\text{-forms on } N\} & \xrightarrow{f^*} & \{\text{exact } k\text{-forms on } M\}. \end{array}$$

It follows that $[f^*\omega]$ is well-defined; (call it $\underline{f^*[\omega]}$).

(exercise: spell out details)

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i.e., there is a contravariant functor $\text{Manifolds}^{op} \xrightarrow{H^k(-)} \text{Vect}_{\mathbb{R}}$

$$M \longmapsto H^k(M)$$

$$(f: M \to N) \longmapsto f^*: H^k(N) \to H^k(M).$$

<u>Key fact</u>: <u>Homotopy invariance</u>

<u>Def</u>: Two maps $\phi_0, \phi_1 : M \longrightarrow N$ are <u>smoothly homotopic</u> if there exists a smooth map $\Phi : M \times [0,1]_t \longrightarrow N$ with $\Phi(-,0) = \phi_0(-)$, $\Phi(-,1) = \phi_1(-)$.

Using the notation $\phi_t := \Phi(-,t)$,

"there exists a smoothly varying family $\{\phi_t\}$ interpolating between ϕ_0 & ϕ_1").

write $\phi_0 \sim \phi_1$ if they are homotopic.

<u>Prop</u>: (Homotopy invariance): Say ϕ_t is a smooth homotopy, $t \in [0,1]$. Then $(\phi_t : M \to N)$

$$\phi_t^* : H^k(N) \to H^k(M) \text{ is } \underline{\text{independent of } t}.$$

(i.e., $\Rightarrow$ if $\phi_0 \sim \phi_1$ then $\phi_1^* \equiv \phi_0^* : H^k(N) \to H^k(M)$).

We'll defer the proof of this briefly & explore computational consequences.