Last time: Homotopy invariance

Def: Two maps $\phi_0, \phi_1 : M \to N$ are smoothly homotopic if there exists a smooth map $\Phi : M \times [0,1]_t \to N$ with $\Phi(-,0) = \phi_0(-)$, $\Phi(-,1) = \phi_1(-)$.

Using the notation $\phi_t := \Phi(-,t)$,

"there exists a smoothly varying family $\{\phi_t\}$ interpolating between ϕ_0 & ϕ_1").

write $\phi_0 \sim \phi_1$ if they are homotopic.

which we haven't yet proven!

Prop: (Homotopy invariance): Say ϕ_t is a smooth homotopy, $t \in [0,1]$. Then $(\phi_t : M \to N)$

$$\phi_t^* : H^k(N) \to H^k(M) \text{ is independent of } t.$$

(i.e., $\Rightarrow$ if $\phi_0 \sim \phi_1$ then $\phi_1^* \equiv \phi_0^* : H^k(N) \to H^k(M)$).

Def: A (smooth) map $\phi : M \to N$ is a homotopy equivalence if $\exists$ a two-sided inverse up to homotopy, i.e., a $\psi : N \to M$ (smooth) with

call ψ "homotopy inverse"

$$\psi \circ \phi \simeq id_M \qquad \phi \circ \psi \simeq id_N.$$

Cor: If $\phi : M \to N$ is a homotopy equivalence then $\phi^* : H^k(N) \to H^k(M)$ is an isomorphism.

Proof: Let $\psi : N \to M$ be a homotopy inverse of ϕ. Then

$$\phi^* \circ \psi^* = (\psi \circ \phi)^* = (id_M)^* = id_{H^k(M)}.$$

$$\psi^* \circ \phi^* = (\phi \circ \psi)^* = (id_N)^* = id_{H^k(N)}. \qquad \boxtimes$$

N.B. Diffeomorphisms are homotopy equivalences, but not nec. vice versa. E.g., there can be homotopy equivalences between manifolds of different dimensions, such as:

Cor: (Poincaré Lemma): $H^k_{dR}(\mathbb{R}^n) = 0$ if $k > 0$. (already know $H^0_{dR}(\mathbb{R}^n) = \mathbb{R}$).

meaning: for $k>0$, every closed k-form is also exact.

Proof: We'll show $\mathbb{R}^n$ is homotopy equivalent to $\mathbb{R}^0 = \{pt.\} = \{0\}$.

By above this will imply $H^k_{dR}(\mathbb{R}^n) \cong H^k_{dR}(\{pt\}) = \begin{cases} 0 & \text{when } k>0. \\ \mathbb{R} & \text{when } k=0. \end{cases}$

Let $\phi : \mathbb{R}^n \to \mathbb{R}^0$, $(x_1,\dots,x_n) \mapsto 0$. $\quad \psi : \mathbb{R}^0 \longrightarrow \mathbb{R}^n$, $0 \longmapsto (0,\dots,0) = \vec{0}$.

Claim: ϕ, ψ are homotopy inverses.

- note $\phi \circ \psi : \mathbb{R}^0 \longrightarrow \mathbb{R}^0 = id_{\mathbb{R}^0}$. ✓
  $0 \mapsto \vec{0} \longrightarrow 0$.

- $\psi \circ \phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n = f_0$ is homotopic to $f_1 = id_{\mathbb{R}^n}$
  $(x_1,\dots,x_n) \mapsto 0 \mapsto \vec{0}$.

  via
  $$f_t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
  $$(x_1,\dots,x_n) \longmapsto (tx_1,\dots,tx_n).$$
  $\left(\begin{array}{l} t=0: \text{ get } f_0 = \psi\circ\phi \\ t=1: \text{ get } id_{\mathbb{R}^n} \end{array}\right)$.

  (note that the map $F: \mathbb{R}^n \times [0,1] \longrightarrow \mathbb{R}^n$, $(x_1,\dots,x_n), t \longmapsto \underbrace{(tx_1,\dots,tx_n)}_{f_t(\vec{x})}$ is smooth).

$\blacksquare$.

So assuming homotopy invariance, we get $H^k_{dR}(\mathbb{R}^n)$

& e.g., $H^k_{dR}(\text{an open ball in } \mathbb{R}^n) \cong H^k(\mathbb{R}^n)$ (b/c $\{\text{open ball in } \mathbb{R}^n\} \underset{\text{diff}}{\cong} \mathbb{R}^n$).

Want to develop more tools for computing $H^k_{dR}(M)$.

Idea: decompose $M = \bigcup U_i$ each U_i open. (and e.g., $U_i \underset{\text{diff.}}{\cong}$ open ball in $\mathbb{R}^n \underset{\text{diff}}{\cong} \mathbb{R}^n$), we understand H^k(pieces); can we reconstruct $H^k(M)$?

Yes!

Mayer-Vietoris Sequence

Let $M = U \cup V$, U, V open sets. Then, we have natural inclusion maps

$$U \xhookrightarrow{i} M = U \cup V$$

$$V \xhookrightarrow{i} M = U \cup V.$$

$$U \cap V \xhookrightarrow{i_U} U$$

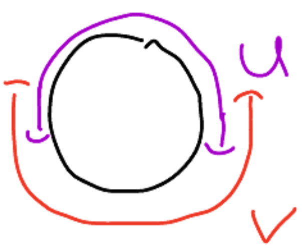
$$U \cap V \xhookrightarrow{i_V} V.$$

which we can alternatively think of as:

$$(*) \quad U \cap V \xrightarrow{i_U, i_V} U \amalg V \xrightarrow{i} U \cup V$$

Example:

S^1.



$M = S^1 = U \cup V$

$U \cong \mathbb{R}$

$V \cong \mathbb{R}$

$U \cap V \cong \mathbb{R} \amalg \mathbb{R}$.

"M-V"

Thm (Mayer-Vietoris sequence):

Given a decomposition of $M = U \cup V$ as above in $(*)$, we obtain the following long-exact-sequence (LES) of cohomology groups:

$$0 \to H^0(M) \xrightarrow{i^*} H^0(U) \oplus H^0(V) \xrightarrow{i_U^* - i_V^*} H^0(U \cap V)$$

$$H^0(U) \oplus H^0(V) = H^0(U \amalg V)$$

$$\xrightarrow{\delta} H^1(M) \xrightarrow{i^*} \overset{H^1(U\sqcup V)}{\overset{\parallel}{H^1(U)\oplus H^1(V)}} \xrightarrow{i_U^*-i_V^*} H^1(U\cap V) \xrightarrow{\delta} \cdots$$

<u>Pf:</u> Later.

Recall that a sequence of vector spaces $\cdots \xrightarrow{f_{i-1}} V^i \xrightarrow{f_i} V^{i+1} \longrightarrow \cdots$

is said to be <u>exact</u> if $\ker f_i = \operatorname{im} f_{i-1}$ for every i.

(<u>long exact sequence</u>: an exact sequence w/ >3 non-zero terms)
LES

<u>Note:</u> (a) $0 \longrightarrow A \xrightarrow{i} B$ exact means $\ker i = \operatorname{im}(0) = 0$. $\Leftrightarrow$ i <u>injectivity</u>

necessarily 0

(b) $A \xrightarrow{j} B \to 0$ exact means j is <u>surjective</u>

(c) $0 \to A \to 0$ exact means $A \cong 0$

(d) $0 \to A \xrightarrow{i} B \to 0$ exact means i is an <u>isomorphism</u> ((a)+(b) apply)

(e) $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ exact means

(<u>short exact sequence</u>, or SES)

- i injective
- j surjective.
- $\ker j = \operatorname{im} i$.

since $\underset{\underset{\displaystyle B/\underbrace{i(A)}_{}}{\parallel}}{B/\ker j} \underset{j}{\cong} \operatorname{im} j = C$

← a subspace isomorphic to A.

Example: $M = S^1$, with U, V as above

$U \cong \mathbb{R}$
$V \cong \mathbb{R}$
$U \cap V \cong \mathbb{R} \sqcup \mathbb{R}$

Compute $H^k_{dR}(M = S^1 = U \cup V)$ using M-V. sequence.

Have (from theorem) a LES:

$\mathbb{R}$ (already know H^0(any M))

$$0 \to \underset{\cong\,\mathbb{R}}{H^0(S^1)} \to \underset{\cong\,\mathbb{R}\oplus\mathbb{R}}{H^0(\mathbb{R}) \oplus H^0(\mathbb{R})} \to \underset{\cong\,\mathbb{R}\oplus\mathbb{R}}{H^0(U \cap V = \mathbb{R} \sqcup \mathbb{R})} \xrightarrow{\delta}$$

$$\to H^1(S^1) \to \underbrace{H^1(\mathbb{R}) \oplus H^1(\mathbb{R})}_{0 \oplus 0} \to \underbrace{H^1(\mathbb{R} \sqcup \mathbb{R})}_{0} \xrightarrow{\delta} H^2(S^1) \to \underbrace{H^2(\mathbb{R}) \oplus H^2(\mathbb{R})}_{0} \to \cdots$$

$= 0$ b/c
$0 \to A \to 0$ is exact
$\Rightarrow A \cong 0$.

- $H^2(S^1) = 0$ by above
- similarly $H^k(S^1) = 0$ for $k \geq 2$ by identical reasoning (using $H^i(\mathbb{R}) = 0$ for $i > 0$).

Rewriting our LES, have:

$$0 \to \underset{\mathbb{R}}{H^0(S^1)} \xrightarrow{i^*} \underset{\cong\,\mathbb{R}\oplus\mathbb{R}\,\cong\,\mathbb{R}^2}{H^0(\mathbb{R}) \oplus H^0(\mathbb{R})} \xrightarrow[(*)]{i_U^* - i_V^*} \underset{\cong\,\mathbb{R}\oplus\mathbb{R}\,\cong\,\mathbb{R}^2}{H^0(\mathbb{R} \sqcup \mathbb{R})} \xrightarrow{\delta} H^1(S^1) \to 0$$

i^* injective by exactness at $H^0(S^1)$.
So $\dim \operatorname{im}(i^*) = 1$.

- exactness at $H^0(\mathbb{R}) \oplus H^0(\mathbb{R})$
  $\Rightarrow \dim \ker(i_U^* - i_V^*) = \dim(\operatorname{im} i^*) = 1$ by above.

- $\ker(*)$ has dimension 1.
  therefore $\dim \operatorname{im}(*) = \dim\left(\text{domain}(*)/\ker(*)\right) = 2 - 1 = 1$. (rank nullity)

- By exactness at $H^0(\mathbb{R} \amalg \mathbb{R})$,

$$\dim \ker \delta = \dim \operatorname{im}(*) = 1.$$

- Rank nullity $\Rightarrow \dim \operatorname{im} \delta = \dim(\text{domain of } \delta) - \dim(\ker \delta)$

$$= 2 - 1 = 1.$$

- exactness $\Rightarrow \operatorname{im} \delta \cong H^1(S^1)$, so $\dim_{\mathbb{R}} H^1(S^1) = 1$.

$\Rightarrow H^1(S^1) \cong \mathbb{R}$.