Homological algebra, culminating in a proof of the M-V exact sequence

Def: A cochain complex $(C^\bullet, d)$ is a sequence of vector spaces C^i, $i \in \mathbb{Z}$, along with maps $d_{(i)}: C^i \to C^{i+1}$

$$\to \cdots \xrightarrow{d_{i-1}} C^i \xrightarrow{d_i} C^{i+1} \xrightarrow{d_{i+1}} C^{i+2} \xrightarrow{d_{i+2}} \cdots$$

Satisfying $d_{i+1} \circ d_i = 0$ for every i. $(*)$

The cohomology of a cochain complex, $H^\bullet(C^\bullet, d)$ is the sequence of vector spaces

$$H^k(C^\bullet, d) := \frac{\ker d_k}{\operatorname{im} d_{k-1}} \qquad \left(\text{note } \operatorname{im} d_{k-1} \subseteq \ker d_k \subseteq C^k \text{ for every } k \text{ by } (*)\right)$$

Example: M manifold, its de Rham cochain complex is:

$$C^i := \Omega^i(M), \qquad d_i := d \text{ exterior differential,}$$

$$\& \quad H^k(\Omega^\bullet(M), d) =: H^k_{dR}(M).$$

Def: A co-chain map $f^\bullet : (C^\bullet, d_C) \to (D^\bullet, d_D)$ is a collection of linear maps $f^i : C^i \to D^i$ for each i making this diagram commute $\forall i$:

$$\begin{tikzcd} C^i \arrow[r, "f^i"] \arrow[d, "d_C"'] \arrow[dr, phantom, "\circlearrowleft"] & D^i \arrow[d, "d_D"] \\ C^{i+1} \arrow[r, "f^{i+1}"'] & D^{i+1} \end{tikzcd} \qquad \text{i.e., } d_D \circ f^i = f^{i+1} \circ d_C$$

A cochain map induces a map $f: H^k(C^\bullet) \to H^k(D^\bullet)$ for every k. sometimes we'll call this map $[f]$ or f_*

(exercise: verify this)

e.g., a smooth map $\phi: M \to N$ induces a co-chain map

$\phi^*: (\Omega^\bullet(N), d) \to (\Omega^\bullet(M), d)$, & hence an induced map
$\phi^*: H^k(N) \to H^k(M)$ for every k.

<u>Def</u>: A <u>short exact sequence</u> (SES) of co-chain complexes, written

$$0 \to C^\bullet \xrightarrow{\phi} D^\bullet \xrightarrow{\psi} E^\bullet \to 0.$$

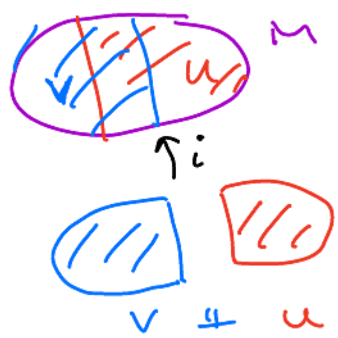
is a pair of cochain maps $\phi:(C^\bullet, d_C) \to (D^\bullet, d_D)$, $\psi: (D^\bullet, d_D) \to (E^\bullet, d_E)$
s.t. for every i, $0 \to C^i \xrightarrow{\phi^i} D^i \xrightarrow{\psi^i} E^i \to 0$ is a <u>short exact sequence</u>.
($\Rightarrow$ ϕ^i injective, ψ^i surjective, and $\operatorname{im}\phi^i = \ker\psi^i$, i.e., ψ^i induces an iso. $D^i/\underbrace{\operatorname{im}(\phi^i)}_{\cong\, C^i} \xrightarrow{\cong} \operatorname{im}\psi^i = E^i$)

<u>Key example</u>:

<u>Prop</u>: Say $M = U \cup V$ where U, V open. Then the following is a SES of chain complexes:

$$(1)\quad 0 \to \underset{= U\cup V}{\Omega^\bullet(M)} \xrightarrow{i^*} \underbrace{\Omega^\bullet(U) \oplus \Omega^\bullet(V)}_{\Omega^\bullet(U \amalg V)} \xrightarrow{i_U^* - i_V^*} \Omega^\bullet(U \cap V) \to 0.$$

where • $i: U \amalg V \to M$.
induced by $U \hookrightarrow M$, $V \hookrightarrow M$
• $i_U: U \cap V \hookrightarrow U$
• $i_V: U \cap V \hookrightarrow V$.



<u>Pf sketch</u>:

$\omega \longmapsto \omega|_U \oplus \omega|_V$.

(i) straightforward verification that $i^*: \Omega^k(M) \longrightarrow \Omega^k(U) \oplus \Omega^k(V)$
is injective. (why? exercise)

(ii) Prove $i_U^* - i_V^*$ is surjective. Let $\omega \in \Omega^k(U \cap V)$

Fix a partition of unity $\{\rho_U, \rho_V\}$ subordinate to $\{U, V\}$

(so $\rho_U + \rho_V \equiv 1$).

Look at $\rho_V|_U \omega \in \Omega^k(U)$ (extend by 0 outside $U \cap V$) &

$-\rho_U|_V \omega \in \Omega^k(V)$ (extend by 0 outside $U \cap V$).

Claim is that $(i_U^* - i_V^*)(\rho_V|_U \omega, -\rho_U|_V \omega)$ is exactly ω.

(check). ✓.

(iii) Exactness at $\Omega^\bullet(U) \oplus \Omega^\bullet(V)$, i.e., that

$$\ker(i_U^* - i_V^*) = \mathrm{im}(i^*).$$

(straightforward exercise).

∎

Prop: (SES $\Rightarrow$ LES)

Given a SES of chain complexes $0 \to C^\bullet \xrightarrow{\phi} D^\bullet \xrightarrow{\psi} E^\bullet \to 0$,

there is always an induced LES of cohomology groups:

$$\cdots \xrightarrow{\delta} H^i(C^\bullet) \xrightarrow{\phi_*} H^i(D^\bullet) \xrightarrow{\psi_*} H^i(E^\bullet) \xrightarrow{\delta} H^{i+1}(C^\bullet) \xrightarrow{\phi_*} H^{i+1}(D^\bullet) \xrightarrow{\psi_*} H^{i+1}(E^\bullet) \xrightarrow{\delta} \cdots$$

<u>Cor</u>: (of above two Props): Proof of the M-V LES:

Given $M = U \cup V$ open cover, the SES (1) induces (by above Prop.)

the M-V LES:

$$\cdots \xrightarrow{\delta} H^k(M) \xrightarrow{i^*} H^k(U)\oplus H^k(V) \xrightarrow{i_U^*-i_V^*} H^k(U\cap V) \xrightarrow{\delta} H^{k+1}(M) \xrightarrow{i^*} H^{k+1}(U)\oplus H^{k+1}(V) \xrightarrow{i_U^*-i_V^*} H^{k+1}(U\cap V) \xrightarrow{\delta} \cdots$$

☒.

<u>Sketch of proof of Prop SES ⇒ LES</u>:

First, we need to construct all the maps in cohomological LES: φ_*, ψ_* defined, what's δ?

$$\delta: H^k(E^\bullet) \longrightarrow H^{k+1}(C^\bullet)$$

$\alpha \in \frac{\ker d_k^E}{\operatorname{im} d_{k-1}^E}$; choose a cycle $e \in E^k$ i.e. $de=0$ & $[e]=\alpha$.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & C^{k+1} \arrow[r, "\varphi_{k+1}"] \arrow[u] & D^{k+1} \arrow[r, "\psi_{k+1}"] \arrow[u] & E^{k+1} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & C^k \arrow[r, "\varphi_k"] \arrow[u, "d_k"] & D^k \arrow[r, "\psi_k"] \arrow[u, "d_k"] & E^k \arrow[r] \arrow[u, "d_k"] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

$$\begin{tikzcd}
\exists\, x \arrow[r, mapsto] & d_k(\tau) \arrow[r, mapsto] & 0 \\
 & \tau \arrow[u, mapsto] \arrow[r, mapsto] & e \arrow[u, mapsto]
\end{tikzcd}$$

choose

$\exists\ \tau \in D^k$ with $\psi_k(\tau)=e$ by surjectivity of ψ_k (not unique!)

Take $d_k^D(\tau) \in D^{k+1}$.

By co-chain map condition, $\psi_{k+1}(d_k^D(\tau)) = d_k^E(\psi_k(\tau)) = d_k^E(e) = 0$..

$\Rightarrow d_k^D(\tau) \in \ker \psi_{k+1} \underset{\text{(by exactness)}}{=} \operatorname{im}(\phi_{k+1})$.

$\Rightarrow \exists!$ element κ with $\phi_{k+1}(\kappa) = d_k^D(\tau)$

(unique by injectivity of ϕ_{k+1}).

<u>want</u>: κ is closed:

note $\phi_{k+2}(d^C\kappa) = d^D(\phi_{k+1}\kappa) = d^D(d^D\tau)$

$= 0$ b/c $(d^D)^2 = 0$.

Since ϕ_{k+2} is injective $\Rightarrow d^C\kappa = 0$.

so κ is closed in C^{k+1}.

<u>Def</u>: $\delta([e]) = [\kappa]$

$[e] \in H^k(D^\bullet)$, $[\kappa] \in H^{k+1}(C^\bullet)$

Need to check this is well-defined? (<u>exercise</u>).

- For a given e, κ depended on a choice of $\tau \in D^k$, $\tau \overset{\psi_k}{\longmapsto} e$.

Claim: if choose a different τ', to get κ' then

$[\kappa] = [\kappa']$, i.e., $\kappa' = \kappa + d(\text{something})$.

- Given a different representative e' of $[e]$ so $e' = e + d(\text{something})$,

need to check outcome is '<u>cohomologous</u>' to κ meaning it differs from κ by something exact. (<u>exercise</u>)

Next time: some words about proof of exactness.