Last time:

Prop: (SES ⇒ LES)

Given a SES of chain complexes $0 \to C^\bullet \xrightarrow{\phi} D^\bullet \xrightarrow{\psi} E^\bullet \to 0$,

there is always an induced LES of cohomology groups:

$$\begin{array}{c} \cdots \\ \xrightarrow{\delta} H^i(C^\bullet) \xrightarrow{(\phi_*)_i} H^i(D^\bullet) \xrightarrow{(\psi_*)_i} H^i(E^\bullet) \\ \xrightarrow{\delta} H^{i+1}(C^\bullet) \xrightarrow{(\phi_*)_{i+1}} H^{i+1}(D^\bullet) \xrightarrow{(\psi_*)_{i+1}} H^{i+1}(E^\bullet) \\ \xrightarrow{\delta} \cdots \end{array}$$

Last time: Defined $\delta: H^i(E^\bullet) \longrightarrow H^{i+1}(C^\bullet)$

$[e] \rightsquigarrow$ pick representative e of $[e]$, pick $\tau \in D^i$ mapping to $e \in E^i$, take $d\tau$, and note that $d\tau = \phi(\eta)$ for some unique $\eta \in C^{i+1}$, & set $\delta([e]) = [\eta]$ w/ $d\eta = 0$.

choices made.

(exercise: well-definedness of δ)

Verification of exactness: we'll just sketch a part of the verification, rest is an exercise.

e.g., want to show $\ker \psi_i = \operatorname{im} \phi_i$ for every i.

- $\ker \psi_i \supset \operatorname{Im} \phi_i$:

  Say $[b] \in \operatorname{Im} \phi_i$, so $[b] = [\phi_i(a)]$ i.e.,

  $b = \phi_i(a) + db'$ where $a \in C^i$, $b' \in D^{i-1}$.

  Then, $\psi_i[b] = \psi_i[\phi_i a + db'] = [\psi_i \phi_i a + \psi_i db']$

  $= [0 + d\psi_i b'] = 0$ ✓.

  b/c $\psi_i \phi_i = 0$ on chain level; b/c ψ_i chain map.

- $\ker \psi_i \subset \operatorname{Im} \phi_i$:

  Say $[b] \in \ker \psi_i$. Pick cocycle rep. b. That means $\psi_i[b] = [\psi_i(b)] = 0$

so $\psi_i(b) = de'$ some $e' \in E^{i-1}$

By short exact sequence on chain level, $\exists\ \beta \in D^{i-1}$ with $\psi_i(\beta) = e'$.

Now, note that $\psi_i(b - d\beta) = de' - d(\psi_i\beta) = de' - de' = 0$,

so $b - d\beta \in \underbrace{\ker \psi_i}_{\text{chain level map}} = \operatorname{im} \phi_i$. So $b - d\beta = \phi_i \alpha$ for some α.

SES of chain complexes.

Now, note that $\phi_i(d\alpha) = d(\phi_i\alpha) = d(b - d\beta) = db - \cancel{d^2\beta}^{\,0} = 0$.

Since ϕ_i injective on chain level (SES of chain cplxes) $\Rightarrow d\alpha = 0$.

$\Rightarrow \phi_i[\alpha] = [\phi_i(\alpha)] = [b - d\beta] = [b]$.

$\Rightarrow [b] \in \operatorname{im}(\phi_i)$ ← cohomology level map (technically write $(\phi_*)_i$).

Exercise: complete these proofs.

Def: Two co-chain maps $\phi_0, \phi_1 : C^\bullet \to D^\bullet$ are chain homotopic if $\exists$ a collection of linear maps $H_i : C^i \to D^{i-1}$ for every i, or $H_\bullet : C^\bullet \to D^{\bullet-1}$, satisfying $(\phi_1 - \phi_0) = dH + Hd$.

$H = \{H_i\}$ is called a chain homotopy between ϕ_0 & ϕ_1.

meaning:

$$(\phi_1)_i - (\phi_0)_i = d_{i-1} \circ H_i + H_{i+1} \circ d_i$$

(as maps $C_i \to D_i$ for each i).

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
C^{i-1} \arrow[r] \arrow[ur, "H_{i-1}"] \arrow[d, "d_{i-1}"'] & D^{i-1} \arrow[d, "d_{i-1}"] \\
C^{i} \arrow[r, "(\phi_0)_i, (\phi_1)_i"] \arrow[ur, "H_i"] \arrow[d, "d_i"'] & D^{i} \arrow[d] \\
C^{i+1} \arrow[r] \arrow[ur, "H_{i+1}"] \arrow[d] & D^{i+1} \arrow[d] \\
 &
\end{tikzcd}$$

Lemma: If $\phi_0, \phi_1 : C^\bullet \to D^\bullet$ are chain homotopic then

$[\phi_0] = (\phi_0)_*$ is equal to $[\phi_1] = (\phi_1)_*$ as maps $H^\bullet(C^\bullet) \to H^\bullet(D^\bullet)$.

Pf: Let $[\alpha] \in H^i(C^\bullet)$. Pick a chain homotopy $H \overset{=}{\{H_i\}}$ between ϕ_0 and ϕ_1.

$$
\begin{aligned}
\text{Then } \quad ((\phi_1)_* - (\phi_0)_*)[\alpha] &= [(\phi_1 - \phi_0)(\alpha)] \\
&= [(dH + Hd)(\alpha)] \\
&= [d(H\alpha) + Hd\alpha] \\
(\alpha \text{ closed}) \quad &= [d(H\alpha)] \\
&= 0. \qquad \blacksquare
\end{aligned}
$$

We want to prove homotopy invariance of deRham cohomology, that is:

if $f_0, f_1 : M \to N$ are (smoothly) homotopic then f_0^* & f_1^* are chain homotopic (as maps $\Omega^\bullet(N) \to \Omega^\bullet(M)$) $\underset{\text{Lemma above}}{\Longrightarrow}$ $f_0^* = f_1^*$ as maps $H^k(N) \to H^k(M)$.

To understand this better, we'll take a digression:

Lie derivatives: M^m manifold

Previously seen that vector fields act on

- functions f or a "0-form" by differentiation:

  given $X \in \mathfrak{X}(M)$, $f \in C^\infty(M) \rightsquigarrow X(f) \in C^\infty(M)$.

- vector fields by bracket:

  given $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(M)$, $\rightsquigarrow [X, Y] \in \mathfrak{X}(M)$.

It turns out vector fields also act on differential forms (& more general tensor fields, such as the vector fields above), by Lie derivative.

Def: Say X vector field on M. We know X determines (at least if M is compact) a global flow (or 1-param. family of diffeos.):

$$\phi_t : M \xrightarrow{\cong} M \qquad \text{satisfying for every } t$$

$$\text{or } \Phi : M \times \mathbb{R} \to M \qquad (d\Phi)_{(p,t)} \underbrace{\left(0, \frac{\partial}{\partial t}\right)}_{\in T_{(p,t)}(M\times\mathbb{R})} = X_{\Phi(p,t)}$$

$$\frac{\partial}{\partial t}(\phi_t(p))$$

(we don't need a global flow, we just need a local flow defined near $t=0$ which always exists).

Each ϕ_t induces $\phi_t^* : \Omega^k(M) \to \Omega^k(M)$, $\phi_0^* = \mathrm{id}_{\Omega^k}$ (for every k).

so we can study, for $\omega \in \Omega^k(M)$,

$$\mathcal{L}_X \omega = \lim_{t\to 0} \frac{\phi_t^*\omega - \underbrace{\omega}_{\phi_0^*\omega}}{t} = \frac{d}{dt}\left(\phi_t^*\omega\right)\Big|_{t=0} \in \Omega^k(M)$$

$$\rightsquigarrow (\mathcal{L}_X\omega)_p = \lim_{t\to 0} \frac{\overbrace{\phi_t^*\omega_{\phi_t(p)} - \omega_p}^{\in \Lambda^k T_p^* M}}{t}.$$

$$\phi_t : p \longmapsto \phi_t(p)$$

$$\Lambda^k(d\phi_t)^* : \Lambda^k T^*_{\phi_t(p)} M \to \Lambda^k T^*_p M$$

This defines $\mathcal{L}_X : \Omega^k(M) \to \Omega^k(M)$ $\forall k$.

(including $k=0$ when $\Omega^0(M) = C^\infty(M)$)

Can also define

$$\mathcal{L}_X : \mathfrak{X}(M) \to \mathfrak{X}(M) \quad \text{by}$$

$$(\mathcal{L}_X Y)_p := \lim_{t\to 0} \frac{d\phi_{-t}(Y_{\phi_t(p)}) - Y_p}{t} = \frac{d}{dt}\left(d\phi_{-t}(Y)\right)\Big|_{t=0}.$$

where $d\phi_{-t} : T_{\phi_t(p)} M \to T_p M$.

b/c $\phi_{-t} : \phi_t(p) \longmapsto p$.

Prop: (omitted from lecture):

(1) $\mathcal{L}_X f = X(f)$, for any $f \in C^\infty(M)$
$\underset{\Omega^0}{\cap}$

(2) $\mathcal{L}_X Y = [X, Y]$ for each $Y \in \mathfrak{X}(M)$.

(3) $\mathcal{L}_X : \Omega^\bullet(M) \to \Omega^\bullet(M)$ and it's a derivation with respect to $\wedge$ which commutes with d. $\longrightarrow$ $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$.

- $\mathcal{L}_X(\alpha \wedge \beta) = (\mathcal{L}_X \alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_X \beta)$

(no sign above, in contrast to d).

Next time: formula for $\mathcal{L}_X$ in terms of d and interior product (to be defined).

+ proof of homotopy invariance.

(which goes roughly by showing that for a homotopy ϕ_t, $\underbrace{\frac{d}{dt}[\phi_t^* \omega]}$ $= 0$ ← on cohomology.
can be related to a Lie derivative in some cases.)