Last time: defined, for $\omega \in \Omega^k(M)$, X vector field.

$\phi_t : M \to M$ flow induced by X (defined at least near $t=0$)

$$\mathcal{L}_X \omega = \lim_{t \to 0} \frac{\phi_t^* \omega - \omega}{t} = \frac{d}{dt}\left(\phi_t^* \omega\right)\Big|_{t=0} \in \Omega^k(M)$$

Lie derivative of X along ω.

- $\mathcal{L}_X f = X(f)$ for $f \in C^\infty(M)$
- can define $\mathcal{L}_X Y$, and $= [X, Y]$
- $\mathcal{L}_X : \Omega^\bullet(M) \to \Omega^\bullet(M)$, is a <u>derivation</u> w.r.t. $\wedge$, commuting with d.

Formula for $\mathcal{L}_X$ in terms of <u>interior product</u> and <u>exterior derivative</u> d

<u>Interior product</u>:

Note: if $\phi \in \Lambda^1 V^* = V^*$, then $i_v \phi = \phi(v) \in \mathbb{R} = \Lambda^0 V^*$

Given V vector space $/\mathbb{R}$, $v \in V$ vector,

define $i_v : \Lambda^k V^* \longrightarrow \Lambda^{k-1} V^*$ by, on pure wedges:

$$f_1 \wedge \cdots \wedge f_k \longmapsto \sum_{\ell} (-1)^{\ell - 1} \underbrace{(f_\ell(v))}_{\in \mathbb{R}} \overbrace{f_1 \wedge \cdots \wedge \hat{f_\ell} \wedge \cdots \wedge f_k}^{\in \Lambda^{k-1} V^*}$$

means look at wedge of all terms other than f_ℓ.

<u>check</u>: this induces a well-defined map $\Lambda^k V^* \to \Lambda^{k-1} V^*$.

(equivalently, the map $(f_1, \ldots, f_k) \longmapsto$ RHS above is an <u>alternating multilinear map</u>).

Now, let X be a vector field on M.

$\rightsquigarrow$ get an operation $i_X : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ "interior product w/ X" "contraction by X"

defined by $(i_X \omega)_p := \overbrace{i_{X_p} \omega_p}^{\in \Lambda^{k-1}(T_pM)^*}$, $X_p \in T_pM$, $\omega_p \in \Lambda^k (T_pM)^*$

satisfying:

(1) For a 1-form ω, $\iota_X\omega = \omega(X)$ e.g., $\omega(X)_p = \omega_p(X_p)$

(2) In general,

$$\iota_X(\alpha\wedge\beta) = (\iota_X\alpha)\wedge\beta + (-1)^{\deg(\alpha)}\alpha\wedge\iota_X\beta.$$

(true b/c it's true pointwise, e.g.,

$$\iota_v(\phi_1\wedge\phi_2) = (\iota_v\phi_1)\wedge\phi_2 + (-1)^{\deg(\phi_1)}\phi_1\wedge\iota_v\phi_2$$

where $\phi_1\in\Lambda^k V^*$, $\phi_2\in\Lambda^l V^*$).

(note: (1)+(2), + compatibility w/ restriction to open sets, determine what ι_X must be).

We can relate ι_X, d, $\mathcal{L}_X$ via:

<u>Prop</u>: [Cartan's "magic" formula]:

For $\omega\in\Omega^*(M)$, $X\in\mathfrak{X}(M)$,

$$\boxed{\mathcal{L}_X\omega = (d\circ\iota_X + \iota_X\circ d)(\omega) = d(\iota_X\omega) + \iota_X(d\omega)} \quad \star$$

<u>Proof idea</u> (Pf is omitted):

<u>Check</u>: $\mathcal{L}_X$ and $d\circ\iota_X+\iota_X\circ d$ both

(i) • are derivations (satisfy Leibniz rule w.r.t. $\wedge$),

(ii) • commute with d.

(iii) • Agree on $\Omega^0(M)$ (note: for $f\in\Omega^0(M)$, $d\circ\iota_X(f)=0$ b/c $\iota_X f\in\Omega^{-1}(M)=0$; so RHS of $\star$ is $\iota_X df = df(X) \underset{\text{(why? check follows from definition)}}{=\!=} X(f) = \mathcal{L}_X f$)

(iv) • are <u>local</u> in the sense that they can be computed (near any point p) in terms of ($\mathcal{L}_X$ & $d\circ\iota_X+\iota_X\circ d$) in coordinate charts.

This reduces to verifying the equality in Euclidean space, where we can use (iii) as base case of an induction in form degree, & use (i) & (ii) to complete the induction. $\boxtimes$.

Returning to de Rham cohomology

First: understand the induced map on cohomology associated to a 1-param. family of diffeomorphisms.

Prop: Let P^p be a (smooth) manifold of dim $= p$, and let

$f_t : P \to P$ be a 1-param. family of diffeomorphisms, $t \in \mathbb{R}$

($\Leftrightarrow$ each f_t is a diffeo., the map $F : (p,t) \longmapsto f_t(p)$ is C^∞,

$f_s \circ f_t = f_{s+t}$, $f_0 = id_P$).

Then, the map $f_t^* : H^k_{dR}(P) \to H^k_{dR}(P)$ is independent of t.

$$\Rightarrow f_t^* = f_0^* = (id_P)^* = id_{H^k(P)}.$$

Pf: Let X on P be the vector field inducing the flow $\{f_t\}$; so

$X_p = \frac{d}{dt}(f_t(p))\big|_{t=0}$, and moreover $X_{f_a(p)} = \frac{d}{dt}(f_t(p))\big|_{t=a}$ for any $a \in \mathbb{R}$.

Roughly we want to show that if $[\omega] \in H^k_{dR}(P)$, then

"$\frac{d}{dt}(f_t^*[\omega])\big|_{t=a} = 0$ for every a." $\Rightarrow$ $f_t^*[\omega]$ is constant in t.

Pick a closed form ω representing a given $[\omega] \in H^k_{dR}(P)$.

We want to study $\frac{d}{dt}(f_t^*\omega)\big|_{t=a}$.

$$= \frac{d}{dt}(f_a^* f_{t-a}^* \omega)\big|_{t=a}$$

$$\underset{u=t-a}{=} \frac{d}{du}(f_a^* f_u^* \omega)\big|_{u=0} = f_a^*\left(\frac{d}{du}(f_u^*\omega)\big|_{u=0}\right)$$

$$= f_a^*(\mathcal{L}_X \omega).$$

So, $\frac{d}{dt}(f_t^*\omega)\big|_{t=a} = f_a^*(\mathcal{L}_X\omega)$.

$$\underset{\text{(cartan's formula)}}{=} f_a^*\left(\left(d\iota_X + \iota_X d\right)(\omega)\right)$$

0 b/c ω closed.

$$= f_a^*\left(d(\iota_X\omega) + \iota_X(\cancel{d\omega})\right)$$

$$= d\left(f_a^*(\iota_X\omega)\right). \qquad (\star\star)$$

for any t_0

Thus, $f_{t_0}^*\omega - \omega = \int_0^{t_0} \left(\frac{d}{dt} f_t^*\omega\right)\Big|_{t=a} da.$

can define by thinking of Riemann integral as limit of Riemann sums.

$$\underset{(\star\star)}{=} \int_0^{t_0} d\left(f_a^*(\iota_X\omega)\right) da.$$

$$\underset{\substack{\text{check}\\ \text{(why?)}}}{=} d\left(\int_0^{t_0} f_a^*(\iota_X\omega)\, da.\right) \quad \text{is exact.}$$

$\Rightarrow [f_{t_0}^*\omega] = [\omega] = [f_0^*\omega].$

$\quad \| $

$f_{t_0}^*[\omega].$

$\Rightarrow f_{t_0}^*$ was independent of t_0. $\boxtimes$

Recall:

$\phi_0, \phi_1 : M \to N$ are smoothly homotopic if $\exists$ smooth $\Phi : M\times[0,1]_t \to N$ with $\Phi(-,0) = \phi_0$, $\Phi(-,1) = \phi_1$.

call Φ a homotopy from $\phi_0 \rightsquigarrow \phi_1$ & sometimes write as $\{\phi_t := \Phi(-,t)\}$

Prop: (Homotopy invariance): Say $\phi_t : M \times [0,1) \to N$ is a homotopy, $t \in [0,1]$.

then $\phi_t^* : H^k_{dR}(N) \to H^k_{dR}(M)$ is independent of t.

(sketch)

Pf: Let $\Phi(-,t) := \phi_t$ be the homotopy. Extend it to a map

$$\Phi : M \times \mathbb{R} \to N \quad \text{satisfying} \quad \begin{cases} \Phi(-,t) = \phi_1 & \text{for } t \geq 1 \\ \Phi(-,t) = \phi_0 & \text{for } t \leq 0. \end{cases}$$

Now, for $\omega \in \Omega^k(N)$, consider $\Omega = \Phi^* \omega \in \Omega^k(M \times \mathbb{R})$

There are inclusion maps

$$\begin{aligned} i_t : M &\hookrightarrow M \times \mathbb{R} \\ m &\longmapsto (m, t) \end{aligned} \qquad \text{for every } t,$$

and note $i_t^* \Omega = i_t^* \Phi^* \omega = \phi_t^* \omega$

$$(\text{b/c } \Phi \circ i_t = \phi_t).$$

Note: there is a 1-param. family of diffeomorphisms

$$\begin{aligned} \Psi_t : M \times \mathbb{R} &\to M \times \mathbb{R}. \\ (m, s) &\longmapsto (m, s+t) \end{aligned} \qquad \left(\text{induced by } X = \frac{\partial}{\partial t}\right).$$

Now, $i_t = \Psi_t \circ i_0$, so on cohomology $i_t^* = i_0^* \circ \Psi_t^*$.

By previous prop, Ψ_t^* is independent of t on cohomology, & $\Psi_t^* = id_{H^k(M \times \mathbb{R})}$

So $i_t^* = i_0^* : H^k(M \times \mathbb{R}) \to H^k(M)$.

$$
\begin{array}{lcccl}
\Rightarrow & i_t^* \circ \Phi^* & = & i_0^* \circ \Phi^* & : H^k(N) \longrightarrow H^k(M). \\
& \| & & \| & \\
& \phi_t^* & = & \phi_0^* &
\end{array}
$$

$\blacksquare$