Some examples of $H^k_{dR}(M)$: (using methods we've developed).

(1) Let M any manifold, I an open interval in $\mathbb{R}$ (or all of $\mathbb{R}$).

WLOG assume I contains 0

Then $\exists$ a (smooth) homotopy equivalence

$$M \times I \simeq M$$

i.e.,

$$\varphi : M \times I \longrightarrow M$$
$$(m,t) \longmapsto m$$

$$\psi : M \longrightarrow M \times I$$
$$m \longmapsto (m, 0)$$

with $\varphi \circ \psi \simeq id_M$ (in fact $= id_M$)

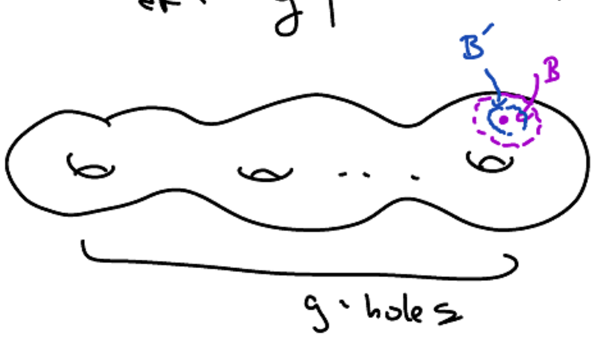
& $\underbrace{\psi \circ \varphi}_{(m,t) \mapsto (m,0)} \simeq id_{M \times I}$ (via homotopy $\{f_s : (m,t) \mapsto (m, st)\}$).

$\Rightarrow H^k_{dR}(M \times I) \cong H^k_{dR}(M) \quad \forall k$.

e.g., $H^k_{dR}(S^1 \times (-\varepsilon, \varepsilon)) \cong H^k_{dR}(S^1) = \begin{cases} \mathbb{R} & k = 0,1 \\ 0 & \text{else.} \end{cases}$

(2) More generally, $E \to M$ any vector bundle then there's a htpy equiv $E \simeq M$, so $H^k_{dR}(E) \cong H^k_{dR}(M)$.

(3) HW exercise: compute $H^k_{dR}(\Sigma_g)$ via (1) studying M-V for

$\Sigma_g =$ 

$U = B$ $(\cong \mathbb{R}^2)$

$V = \Sigma_g \setminus \overline{B'}$

$U \cap V = S^1 \times (-\varepsilon, \varepsilon)$

$(\simeq S^1$ by above$)$

(ii) understand $\Sigma_g \setminus \overline{B}'$ as a "bouquet of $2g$ bands"

e.g., $\Sigma_1 \setminus \overline{B}'$ looks like.

Inductively apply M-V, e.g.,

eventual outcome: $$H^k_{dR}(\Sigma_g) = \begin{cases} \mathbb{R} & k=0,2 \\ \mathbb{R}^{2g} & k=1 \\ 0 & \text{else} \end{cases}$$

There are various numerical invariants one can extract from $H^\bullet_{dR}(M)$ e.g.,

$$b_i(M) := \dim_{\mathbb{R}} H^i_{dR}(M) \qquad \underline{i^{th} \text{ Betti number}}$$

Another invariant with particular significance: <u>Euler characteristic</u>

$$\chi(M) = \sum_{i=0}^{m} (-1)^i \dim H^i_{dR}(M) = \sum_{i=0}^{M} (-1)^i b_i(M).$$

<u>ex</u>: (i) $\chi(\mathbb{R}^n) = 1$

(ii) $\chi(S^n) = 1 + (-1)^n = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$

(iii) $\underset{\Sigma_1}{\underset{\shortparallel}{\chi(T^2)}} = 1 - 2 + 1 = 0.$

(iv) more generally $\chi(\Sigma_g) = 2 - 2g$.

For cpct. M^2, $\chi \underset{\text{Thm.}}{\overset{\uparrow}{=}} V - E + F$, where $V = \#$vertices, $E = \#$edges, $F = \#$faces of any triangulation of M^2.

<u>Forms & deRham coh. w/ compact support.</u>

Given a vec. bundle $\begin{matrix} E \\ \pi \downarrow \\ M \end{matrix}$, we define $\Gamma(E) = $ sections of $E = \{ s: M \to E,\ \pi \circ s = id_M \}$

Can define $\Gamma_c(E) \subseteq \Gamma(E)$ compactly supported sections

$$\Gamma_c(E) = \{ s \in \Gamma(E) \mid s(p) = (p, \overset{E_p}{s_p}) = (p, 0) \text{ outside a compact set in } M \}$$

(in other words $\Gamma_c(E)$ = sections whose support $\subseteq$ a cpct. subset of M).

If M cpct, then $\Gamma_c(E) = \Gamma(E)$.

$\rightsquigarrow$ since $\Omega^k(M) = \Gamma(\Lambda^k T^*M)$,

can define $\Omega^k_c(M) = \Gamma_c(\Lambda^k T^*M)$ compactly supported k-forms.

Lem: (exercise): $d : \Omega^k(M) \to \Omega^{k+1}(M)$ preserves the condition of having cpct. support

e.g., $(\Omega^\bullet_c(M), d)$ is a co-chain complex.

Def: The kth deRham coh. group with compact support of M

is $H^k_{c(,dR)}(M) := H^k(\Omega^\bullet_c(M), d)$

$$= \frac{\ker d_k : \Omega^k_c(M) \to \Omega^{k+1}_c(M)}{\operatorname{im} d_{k-1} : \Omega^{k-1}_c(M) \to \Omega^k_c(M)}$$

M cpct. $\Rightarrow H^k_c(M) = H^k(M)$.

But this is not nec. true if M is non-cpct!

E.g.,

Lem: If M is non-compact and connected, then $H^0_c(M) = 0$. $(\neq \mathbb{R} = H^0(M))$!

Pf: $H^0_c(M) = \ker d : \Omega^0_c(M) \to \Omega^1_c(M)$

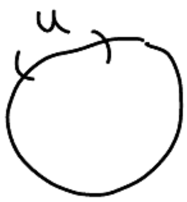
$$\Omega^0_c(M) = C^\infty_c(M).$$

We've prev. seen for any $f \in C^\infty(M)$, if $df = 0$ then f is locally constant. If $f \in C^\infty_c(M)$, it follows that f must vanish somewhere $\underset{(M \text{ connected})}{\Longrightarrow}$ $f \equiv 0$. ☒,

Properties:

If $f : M \to N$ then note $f^* : \Omega^k(N) \to \Omega^k(M)$ need to preserve the condition of having compact support!

(e.g., $\mathbb{R} \xrightarrow{f} \{0\}$, $g : \{0\} \to \mathbb{R}$ constant fn. w/ value 5 $\Rightarrow$ $f^*g = 5$ not cptly supported)

(e.g., 

M cpct manifold, $U \hookrightarrow M$ open inclusion, pull back = restriction need not preserve cpt. support).

Additional hypothesis:

Recall $f : M \to N$ is proper if $\forall$ cpct. $K \subseteq N$, $f^{-1}(K)$ is cpct. in M.

If f is proper, $f^*(\Omega^k_c(N)) \subseteq \Omega^k_c(M)$, so

$$f^* : (\Omega^\bullet_c(N), d) \longrightarrow (\Omega^\bullet_c(M), d) \quad \text{co-chain map}$$

$$\Rightarrow f^* : H^k_c(N) \to H^k_c(M).$$

If $f \simeq g$ via a proper homotopy, that is $\exists$ $F : M \times [0,1] \longrightarrow N$ which is proper & $F(-,0) = f$, $F(-,1) = g$, then same arguments as before show $f^* = g^* : H^k_c(N) \to H^k_c(M)$.

(warning: $\exists$ f, g proper maps which are homotopic, but not properly homotopic, with $f^* \neq g^*$ on $H_c^\bullet$).

(e.g., $\mathbb{R} \underset{h.e.}{\simeq} 0$ but not via proper homotopies / maps; & indeed $H_c^0(\mathbb{R}) = 0 \neq H_c^0(0) = H^0(0) = \mathbb{R}$.),

Note: Let $U \overset{i}{\hookrightarrow} M$ inclusion of an open set.

Then, can define

$$i_! : \Omega_c^k(U) \to \Omega_c^k(M) \quad \text{(covariant functoriality!)}$$

by $\alpha \longmapsto (i_! \alpha)_p := \begin{cases} \alpha_p & p \in U \\ 0 & p \notin U \end{cases}$

(check $i_! \alpha$ is smooth b/c α has cpct. support).

gives a cochain map & therefore

$$i_! : H_c^k(U) \to H_c^k(M).$$

Ex: $$H_c^1(\mathbb{R}) = \frac{\ker d_1 : \Omega_c^1 \to \cancel{\Omega_c^2}\ 0}{\operatorname{im} d_0 : \Omega_c^0 \to \Omega_c^1} = \frac{\Omega_c^1}{d_0(\Omega_c^0)}$$

$$\Omega_c^1(\mathbb{R}) \cong C_c^\infty(\mathbb{R})$$

$(\alpha = f\,dx) \longleftrightarrow f.$

with respect to which

$$\begin{array}{ccc} f & \longmapsto & \frac{df}{dx}dx \\ d: \Omega_c^0(\mathbb{R}) & \longrightarrow & \Omega_c^1(\mathbb{R}) \\ \wr\wr & & \wr\wr \\ C_c^\infty(\mathbb{R}) & \longrightarrow & C_c^\infty(\mathbb{R}) \\ f & \longmapsto & f' \end{array}$$

Exercise: check: $H^1_c(\mathbb{R}) \xrightarrow{\cong} \mathbb{R}$ (& prev. shown $H^0_c(\mathbb{R}) = 0$).

Note: $\exists$ map $\int_{\mathbb{R}} (-)dx : C^\infty_c(\mathbb{R}) \longrightarrow \mathbb{R}$.

- surjective
- if f is cptly supported, say $\operatorname{supp} f \subseteq [-R, R]$

$$\int_{\mathbb{R}} f' dx = \int_{-R}^{R} f' dx \underset{FTC}{=} f(R) - f(-R)$$
$$= 0.$$

$$\Rightarrow \int_{\mathbb{R}} (-) : \Omega^1_c(\mathbb{R}) \big/ d(\Omega^0_c(\mathbb{R})) \longrightarrow \mathbb{R}$$

Exercise: show injective; i.e., show if $\int_{\mathbb{R}} \underset{\text{cptly supported}}{g}\, dx = 0$,

then $g = f'$ for some $f \in C^\infty_c(X)$.

($f(x) = \int_{-R}^{x} g(u)du$ for some $R << 0$ s.t., $\operatorname{supp} g \subseteq [-R, R]$)

This has cpct. support if g has cpct support & $\int_{\mathbb{R}} g\, dx = 0$).

Next time: integration of differential forms & study $\int$ as a map on de Rham cohomology.