Today: Integration of differential forms

cpct. support (vacuous if M cpct.)

Goal: Define, for M^m an oriented manifold, and $\omega \in \Omega^m_c(M)$, an integral $\int_M \omega$.

depends on orientation of M up to sign.

In $\mathbb{R}^m$, have Riemann integral, roughly defined as follows:

$$R = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^m \quad \text{"rectangle"}$$

Consider $f: R \longrightarrow \mathbb{R}$

OR $f: U \longrightarrow \mathbb{R}$ (U open $\subset \mathbb{R}^m$) fxn. w/ cpct. support.

"upper Riemann sum of f assoc. to partition"

In the first case, define $\int_R f\, dx_1 \cdots dx_n = \lim\limits_{P \text{ partitions of } R} U(f, P) = \lim\limits_{P \text{ partitions of } R} L(f, P)$

"lower Riemann sum"

"f integrable" $\Rightarrow$ these limits exist and are equal, & therefore $\int_R$ well-defined.

Given a partition $P = (P_1, \ldots, P_k)$, (rectangles in R)

$$U(f, P) = \sum \mathrm{vol}(P_i) \cdot \sup(f|_{P_i})$$

$$L(f, P) = \sum \mathrm{vol}(P_i) \cdot \inf(f|_{P_i})$$

If f is continuous $\Rightarrow$ it's integrable (in particular, smooth f's are integrable).

In second case $f: U \to \mathbb{R}$ w/ cpct. support, choose $R \supset \operatorname{supp} f$, & extend f to $\bar{f}: R \to \mathbb{R}$ (extension necessary if $R \not\subset U$) by 0, i.e., $\bar{f}(r) = 0$ if $r \notin \operatorname{supp} f$.

& define $$\int_U f\, dx_1 \cdots dx_n := \int_R \bar{f}\, dx_1 \cdots dx_n.$$ (check independent of choice of R).

Change of variables formula:

In 1D:

If $g: [a,b] \to [c,d]$ smooth diffeo, then

$$\underbrace{\int_{g(a)}^{g(b)} f\,dx}_{} = \int_a^b (f\circ g)\,g'\,dx.$$

$$= \begin{cases} \int_c^d f\,dx & \text{if } g \text{ orientation preserving} \\ & (\text{so } g'>0) \\ -\int_c^d f\,dx & \text{if } g \text{ orientation reversing.} \\ & (\text{so } g'<0) \end{cases}$$

$\Longrightarrow$ can rewrite as follows:

$$\boxed{\int_{g([a,b])} f\,dx = \int_a^b f\circ g\,|g'|\,dx.}$$

$g([a,b])$ means $\int_c^d$

$U \underset{\text{open}}{\subseteq} \mathbb{R}^m_{x_1,\dots,x_m}$

$\phi \big\downarrow \cong$

$V \underset{\text{open}}{\subseteq} \mathbb{R}^m_{y_1,\dots,y_m}$

More generally, if U, V open sets in $\mathbb{R}^m$ w/ coordinates $(x_1,\dots,x_m)$, $(y_1,\dots,y_m)$ & $\phi: U \xrightarrow{\cong} V$ diffeomorphism, & $f: V \to \mathbb{R}$ w/ cpct. support, then

$$\boxed{\int_V f(y)\,dy_1\cdots dy_m = \int_U f(\phi(x))\,|d\phi|\,dx_1\cdots dx_m}$$

$\phi = (\phi_1,\dots,\phi_m)$

shorthand for $|\det(d\phi)|$.

abs. value of determinant of the total deriv/ Jacobian matrix

$$d\phi = \left[\,\partial\phi_i/\partial x_j\,\right]_{i,j}.$$

Integration of forms

Consider first $V \underset{\text{open}}{\subseteq} \mathbb{R}^m_{y_1,\dots,y_m}$ & $\omega \in \Omega^m_c(V)$, so $\underline{\omega = f\,dy_1\wedge\cdots\wedge dy_m}$.

Proposal is to define $\int_V \omega := \int_V f\,dy_1\cdots dy_m$.

Diffeomorphism invariant? Say $\phi: U \xrightarrow{\cong} V$, then

$$\int_U \phi^*\omega \overset{??}{=} \int_V \omega$$

$\phi = (\phi_1,\dots,\phi_m)$

$d\phi_i = \sum \frac{\partial\phi_i}{\partial x_j}\,dx_j$

Note $\int_U \phi^*\omega = \int_U (f\circ\phi)\,d\phi_1\wedge\cdots\wedge d\phi_m$

$$= \int_U (f\circ\phi)\det(d\phi)\,dx_1\wedge\cdots\wedge dx_m$$

$$:= \int_U (f\circ\phi)\det(d\phi)\,dx_1\cdots dx_m$$

$$\overset{\text{equality only holds if } \det(d\phi)=|\det(d\phi)|,\text{ i.e., if } \det(d\phi)>0 \text{ everywhere.}}{=\!=} \int_V f\,dy_1\cdots dy_m = \int_V \omega$$

change of variables formula

equality only holds if $\det(d\phi) = |\det(d\phi)|$, i.e., if $\det(d\phi) > 0$ everywhere.

i.e., $\int_V \omega = \int_U \phi^*\omega$ if $\phi: U \xrightarrow{\cong} V$ was orientation preserving.

(if it does, say M orientable)

Recall: An orientation on M^m (if exists), e.g., a choice of component of $\Lambda^m TM\setminus\{0\}$ or a component of $\Lambda^m T^*M\setminus\{0\}$ or an equiv. class of nowhere vanishing top form up to pos. scaling, induces a (maximal) oriented atlas $\mathcal{A}_{or(max)} \subseteq \mathcal{A}_{(max)}$ consisting of those

(U,ϕ) s.t. $\underset{\underset{M}{\cap\text{ open}}}{\phi: U} \xrightarrow{\cong} \underset{\underset{\mathbb{R}^m}{\cap\text{ open}}}{\phi(U)}$ is orientation preserving (for std orientation on $\mathbb{R}^m$ & one on U induced by chosen orientation on M)

For any pair of such (U,ϕ), (V,ψ), the transition fns.

$$\psi\circ\phi^{-1}: \underset{\underset{\mathbb{R}^m}{\cap\text{ open}}}{\phi(U\cap V)} \xrightarrow{\cong} \underset{\underset{\mathbb{R}^m}{\cap\text{ open}}}{\psi(U\cap V)}$$

is an oriented diffeo. (so $\det(d(\psi\circ\phi^{-1})) > 0$)

Thm: Given an oriented M^m, $\exists$ a unique linear map

$$\int_M(-): \Omega^m_c(M) \longrightarrow \mathbb{R}.$$

such that for any $\omega\in\Omega^m_c(M)$ with $\mathrm{supp}(\omega)\subset U_\alpha$, w/ $(U_\alpha,\phi_\alpha)\in\mathcal{A}_{or,max}$,
& $(\phi_\alpha^{-1})^*\omega = f_\alpha dx_1\wedge\cdots\wedge dx_m \in \Omega^m_c(\phi_\alpha(U_\alpha))$,
then $\int_M\omega = \int_{\phi_\alpha(U_\alpha)}(\phi_\alpha^{-1})^*\omega = \int_{\phi_\alpha(U_\alpha)} f_\alpha\,dx_1\cdots dx_m$. (A)

Proof start/sketch: Let $\omega \in \Omega^m_c(M)$

(i) • If $\operatorname{supp}\omega \subseteq U_\alpha$, where $(U_\alpha, \phi_\alpha) \in \mathcal{A}_{or,max}$. Then, we use $\phi_\alpha : U_\alpha \xrightarrow{\cong} \phi_\alpha(U_\alpha)$ to define $\int_M \omega$ as in $(*)$.

(ii) • More generally, pick an oriented atlas $\mathcal{A}_{or} \subseteq \mathcal{A}_{or,max}$ which is locally finite & $\mathcal{A}_{or} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$, & let $\{\psi_\alpha\}_{\alpha\in I}$ be a partition of unity subordinate to U_α.

Then $\omega = \sum_\alpha \psi_\alpha \omega$; but now each $\psi_\alpha \omega$ is supported in U_α, hence $\int_M \psi_\alpha \omega$ is already defined above, now linearity forces us to define

$$\int_M \omega = \sum_\alpha \int_M \psi_\alpha \omega = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^* (\psi_\alpha \omega).$$

well-defined (finite) sum over α by cpct. support of ω; restrict to finitely many U_α covering $\operatorname{supp}(\omega)$.

Exercise: check that this proposed definition is well-defined i.e., doesn't depend on choice of oriented atlas made or choice of $\{\psi_\alpha\}$ subordinate to U_α.

(e.g., in (i), say $\operatorname{supp}\omega \subset (U_\alpha, \phi_\alpha)$ and $\operatorname{supp} \subset (U_\beta, \phi_\beta)$.

Then $\int_M \omega = \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^* \omega = \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^* \phi_\beta^* (\phi_\beta^{-1})^* \omega$

$= \int_{\phi_\alpha(U_\alpha)} (\phi_\beta \circ \phi_\alpha^{-1})^* (\phi_\beta^{-1})^* \omega$

vs. $\int_M \omega = \int_{\phi_\beta(U_\beta)} (\phi_\beta^{-1})^* \omega$

b/c $\det(\phi_\beta \circ \phi_\alpha^{-1}) > 0$

by oriented condition

$\phi_\beta \circ \phi_\alpha^{-1}$ oriented <u>diffeo.</u> by hypothesis