- Final HW due Monday, further extensions OK.
- In-class final exam next week (5/4), 11am-1pm (double check!) in person only (no Zoom), this room KAP 137.

- look at old G&T quals (USC math website)
- problems in Lee's book

Last time: defined

$$\int_M \omega \in \mathbb{R} \quad \text{for } M^m \text{ oriented manifold and } \omega \in \Omega_c^m(M) \text{ compactly supported top form}$$

What to do with $\omega \in \Omega_c^p(M)$ for $p < \dim M$? Integrate along submanifolds.

Def: If $\omega \in \Omega_c^p(M)$ and $P^p \overset{i}{\hookrightarrow} M$ is a submanifold, oriented s.t. $i^*\omega$ $(\omega|_P)$ is also compactly supported (e.g., if i is proper)

$$\int_P \omega := \int_P i^*\omega \left(= \int_P \omega|_P\right). \in \mathbb{R}.$$

Important case for us: if M^m is a cpct oriented manifold-with-boundary ∂M, and $\omega \in \Omega^{m-1}(M)$, we can take $\int_{\partial M} \omega := \int_{\partial M} i^*\omega. \left(= \int_{\partial M} \omega|_{\partial M}\right)$ $\boxed{\text{Q: why is } \partial M \text{ oriented?}}$

Recall:

- In a m'fld-with-boundary, the smooth atlas consists of charts locally homeo. to open subsets of $\mathbb{H}^m = \{x_1 \leq 0\} \subseteq \mathbb{R}^m$, s.t. transition functions are smooth.

- $\partial M :=$ the set of points of M which map $\partial \mathbb{H}^m = \{x_1 = 0\}$ under some, equivalently any, chart map ($m-1$ dim'l m'fld).

∂M is a well-defined smooth manifold dim. $m-1$ & $M \setminus \partial M$ is a smooth manifold of dim. m.

Ex: • $B^n = \{\sum_{i=1}^n x_i^2 \leq 1\} \subseteq \mathbb{R}^n$ is a m'fld-with-boundary, & $\partial B^n = S^{n-1}$.

• $f: M^m \to \mathbb{R}$ smooth, $a \in \mathbb{R}$ regular value $\Rightarrow Q = f^{-1}((-\infty, a])$ is a m-dim'l manifold with boundary, $\partial Q = f^{-1}(a)$ ($m-1$ dim'l m'fld).

Prop: If M is an orientable manifold-with-boundary, then ∂M is orientable. Moreover, any orientation of M induces an orientation of ∂M.

Proof sketch:

Method 1: Let $\{(U_\alpha, \phi_\alpha)\}$ be an oriented atlas for M, meaning we fix an orientation

$\sigma_\alpha \ (\overset{\text{e.g.}}{=} \mathrm{std}_{\mathbb{R}^m}) \in \mathrm{or}(\phi_\alpha(U_\alpha))$ and each $\phi_\alpha \circ \phi_\beta^{-1}$ sends $\underset{(\overset{\text{"}}{\mathrm{std}}_{\mathbb{R}^m})}{\sigma_\beta}$ to $\underset{(\overset{\text{"}}{\mathrm{std}}_{\mathbb{R}^m})}{\sigma_\alpha}$.

$\quad \overset{\shortparallel}{[(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m})]}$

Then, on $\partial \phi_\alpha(U_\alpha)$ fix the orientation $\partial\sigma_\alpha$ defined as follows:

if $\sigma_\alpha = \mathrm{std}_{\mathbb{R}^m}$ on $\underset{\mathbb{H}^m}{\overset{\cup}{\phi_\alpha(U_\alpha)}}$ then $\partial\sigma_\alpha = \mathrm{std}_{\mathbb{R}^{m-1}} = [(\frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_m})]$ on $\phi_\alpha(U_\alpha) \cap \partial\mathbb{H}^m \subseteq \partial\mathbb{H}^m$

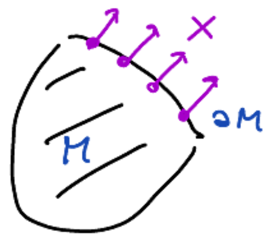
otherwise, can always write $\sigma_\alpha = [(\frac{\partial}{\partial x_1}, \text{something} \qquad)] \Rightarrow$ define $\partial\sigma_\alpha := [(\text{something})]$.

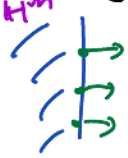
equiv. class of a basis of tangent vectors w/ $\frac{\partial}{\partial x_1}$ first, & something is a basis of $T_p\partial\mathbb{H}$.

exercise.

Method 2:

Claim: For any manifold-with-boundary M, $\exists$ a non-zero outward pointing vector field along the boundary $X \in \Gamma(TM|_{\partial M})$.

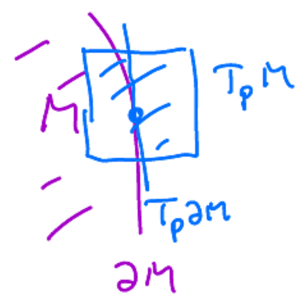


(partition of unity argument: on a given chart $\mathbb{H}^m$, consider $\frac{\partial}{\partial x_1}$  , then

patch together using the partition of unity (so $X = \sum \psi_\alpha \left(d(\phi_\alpha^{-1})\left(\frac{\partial}{\partial x_1}\right)\right)$) outward pointing vector field on U_α.

partition of unity subordinate to $\{(U_\alpha, \phi_\alpha)\}$)

Using this: let $\omega \in \Omega^m(M)$ be nowhere vanishing top form representing given orientation $[\omega] \in \mathrm{or}(M)$.



Restrict ω pointwise to ∂M, so $\omega|_{\partial M} \in \Gamma(\partial M, \Lambda^m T^*M|_{\partial M})$.

Pick X_{outward} outward pointing vector field as in claim, and consider

$$\iota_{X_{\text{outward}}} \omega|_{\partial M} \in \Gamma(\partial M, \Lambda^{m-1} T^*\partial M) = \Omega^{m-1}(\partial M).$$

claim: nowhere vanishing (uses ω, X nowhere vanishing and $X \pitchfork T\partial M$). exercise:

note: at $p \in \partial M$, $\iota_{(X_{\text{outward}})_p} \omega_p \in \Lambda^{m-1} T_p^*M$, now further restrict RHS to $\Lambda^{m-1} T_p^*\partial M$ above.
($T_p\partial M \hookrightarrow T_pM$ induces $T_pM^* \to T_p\partial M^*$)

Stokes' Theorem :

Thm: Let ω be an $(m-1)$-form on an oriented manifold-with-boundary, M^m, with compact support (a vacuous condition if M, ∂M cpct.). Then,

$$\int_M d\omega = \int_{\partial M} \omega$$

w/ induced boundary orientation (as induced by Prop.).

Rmks:
- ∂ happily switches places "jumps up or down"
- This generalizes
  - fundamental theorem of calculus — how? $\int_{[a,b]} df = \int_{\{a\}^-, \{b\}^+} f = f(b) - f(a)$; $\int_a^b f'(x)\,dx$.
  - Green's theorem in 2D
  - Stokes' & divergence theorems in 3D

  (how? need to relate $d(-)$ to div & curl; we've seen a bit of how to do this).

Cor: Let M be an oriented m-dim'l manifold (without boundary, 'closed'), $\alpha \in \Omega_c^{m-1}(M)$. Then

$$\int_M d\alpha \overset{\text{stokes}}{=} \int_{\partial M = \emptyset} \alpha = 0 \quad \text{for all } \alpha \in \Omega_c^{m-1}(M).$$

i.e., $\int_M (\text{exact forms in } \Omega_c^m(M)) = 0$

note: any $\omega \in \Omega_c^m(M)$ is closed, b/c $\Omega_c^{m+1}(M) = \{0\}$ $(\dim M = m)$ so $\int_M(-)$: closed m-forms $\to \mathbb{R}$, sends exact forms to 0.

Cor: For an oriented M^m (w/o ∂), $\exists$ a well-defined map

$$\int_M(-) : H_c^m(M) \longrightarrow \mathbb{R}$$

$H_c^m(M) = \dfrac{\text{~~closed~~ cptly supported } m\text{-forms}}{\text{exact cptly supported } m\text{-forms}}$

$$[\omega] \longmapsto \int_M \omega.$$

well-defined? If $[\omega] = [\omega']$, so $\omega = \omega' + d\kappa$ then $\int_M \omega = \int_M \omega' + \cancel{\int_M d\kappa}$ (0 by above). ✓.

($\Rightarrow$ if $\int_M \omega \neq 0$, ω cannot be exact!).

Cor: (detecting non-vanishing cohomology classes of other degrees). Given a closed form $\alpha \in \Omega_c^p(M)$, and a submanifold $P \subseteq M$ which is oriented, properly embedded (so $\alpha|_P \in \Omega^p_c(P)$),

$\Rightarrow$ if $\int_P \alpha \neq 0$ then α is not exact (& $[\alpha] \neq 0$ in $H^p_c(M)$). (why? b/c we've shown $H^p_c(M) \xrightarrow{i^*} H^p_c(P) \xrightarrow{\int_P} \mathbb{R}$, $[\alpha] \longmapsto [i^*\alpha] \longmapsto$ non-zero.)

(also, for $\alpha \in \Omega^p(M)$, $P \subseteq M$ compact oriented manifold so $\alpha|_P \in \Omega^p(P) = \Omega^p_c(P)$, if $\int_P \alpha \neq 0$ then $[\alpha] \neq 0$ in $H^p(M)$).

Ex: Let $M^2 = \mathbb{R}^2 \setminus 0$ with one-form $\alpha = \frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy$. ($\alpha = $ "$d\theta$", "$\theta = \tan^{-1}\left(\frac{y}{x}\right)$", but be careful b/c θ not well-defined).

- check α is closed (exercise - compute directly $d\alpha$). Want to see $[\alpha] \neq 0$ in $H^1_{dR}(M)$.
- Let $S^1 \hookrightarrow \mathbb{R}^2 \setminus 0$ unit circle. ($x^2+y^2=1$), w/ orientation as ∂B^2, using σ_{std} on B^2.

Compute that $\int_{S^1} \alpha \underset{x^2+y^2=1}{=} \int_{S^1} y\,dx - x\,dy \underset{\text{Stokes}}{=} \int_{B^2} d(y\,dx - x\,dy) = \int_{B^2} -2\,dx \wedge dy$

$\{x^2+y^2 \leq 1\}$

this is restriction to S^1 of the form $y\,dx - x\,dy$ on B^2.

$= -2\,\mathrm{vol}(B^2)$

$= \boxed{-2\pi} \neq 0$.

$\Rightarrow$ α not exact, so $[\alpha] \neq 0$ in $\mathbb{R}^2 \setminus 0$.

(if α were exact then $\alpha|_{S^1}$ would be exact, hence $\int_{S^1} \alpha|_{S^1} = 0$).

Proof sketch of Stokes' theorem: (by reduction to FTC, eventually).

Two special cases: $M = (0,1)^m$ and $M = (0,1] \times (0,1)^{m-1}$

Stokes' in these cases:

(i) For $\alpha \in \Omega^{m-1}_c((0,1)^m)$, $\int_{(0,1)^m} d\alpha = 0$

(ii) For $\beta \in \Omega^{m-1}_c((0,1] \times (0,1)^{m-1})$, $\int_{(0,1] \times (0,1)^{m-1}} d\beta = \int_{\{1\} \times (0,1)^{m-1}} \beta$.

Can treat (i) + (ii) uniformly by extension by 0 to $[0,1]^m$ (technically this is a manifold-with-corners, but a version of Stokes' theorem applies in such cases too; we'll just need to know how it works for $[0,1]^m$). More generally, have:

Prop

For $K \in \Omega^{m-1}([0,1]^m)$, $\int_{[0,1]^m} dK = \underbrace{\sum_i \left(\int_{[0,1] \times \cdots \times 1 \times \cdots \times [0,1]} K - \int_{[0,1] \times \cdots \times 0 \times \cdots \times [0,1]} K \right)}_{\star}$.

ith $\quad$ ith.

This subsumes (i) and (ii) because, e.g., in (ii), the extension of β to $[0,1]^m$ satisfies $\beta|_{0\times[0,1]^{m-1}} = 0$ and $\beta|_{[0,1]\times\cdots\underset{\wedge}{0}\times\cdots[0,1]} = 0$

Pf sketch:

Denote by $dx_{\hat{i}} := dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_m \in \Omega^{m}([0,1]^m)$.

If $\gamma = \sum_i f_i\, dx_{\hat{i}}$, then

only bit of df_i that survives when $\wedge$ with $dx_{\hat{i}}$

- $d\gamma = \sum_i \overbrace{\frac{\partial f_i}{\partial x_i}dx_i}\, \underbrace{\wedge\, dx_{\hat{i}}}_{\substack{\shortparallel \\ (-1)^i\, dx_1\wedge\cdots\wedge dx_m.}}$

$\star\star$
$$\begin{cases} \cdot\ \gamma\big|_{[0,1]\times\cdots\times 0\times\cdots[0,1]} = f_i\big|_{x_i=0}\, dx_{\hat{i}} \\ \cdot\ \gamma\big|_{[0,1]\times\cdots\times 1\times\cdots\times[0,1]} = f_i\big|_{x_i=1}\, dx_{\hat{i}} \end{cases}$$

(note: $dx_{\hat{j}}\big|_{[0,1]\times\cdots\times \overset{\text{or } 1}{0}\times\cdots\times[0,1]} = 0$ if $j\neq i$).
b/c $dx_i|_{x_i=0 \text{ or } 1} = d(\text{constant}) = 0$, & $dx_{\hat{j}}$ contains dx_i)

Now $\int_{[0,1]^m} d\gamma = \sum_i \int_{[0,1]^{m-1}}\left(\int_0^1 \frac{\partial f_i}{\partial x_i}dx_i\right)dx_1\cdots\widehat{dx_i}\cdots dx_m$

Apply FTC & compare to RHS $\star$, using $\star\star$. ∎