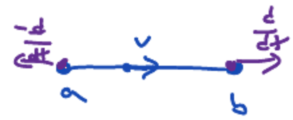
Stokes' theorem:

M^m oriented manifold-with-boundary, $\omega \in \Omega_c^{m-1}(M)$.

Then $\int_M d\omega = \int_{\partial M} \omega$ ← with orientation induced by M, as in Monday's lecture

Rmk about boundary orientation when $m=1$:



std. orientation on $[a,b]$ given by $[\omega = dt]$; note if v is a tangent vector as above $dt(v) > 0$.

at b boundary, an outward pointing v.f. is $\frac{d}{dt}$, and $\iota_{\frac{d}{dt}} \omega = dt(\frac{d}{dt}) = 1 > 0$. so induced orientation on $\{b\}$ is "+". (Recall $or(pt.) = \{+, -\}$).

At $\{a\}$, an outward pointing vec. field is $-\frac{d}{dt}$, and $\iota_{-\frac{d}{dt}} \omega = dt(-\frac{d}{dt}) = -1 < 0$ so induced boundary orientation on a is "−".

Pf sketch:

- True for $M = (0,1)^m$ or $(0,1] \times (0,1)^{m-1}$, any $\omega \in \Omega_c^{m-1}(M)$ (last time).

- For general M, find a locally finite oriented atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ where for each α, either

  (i) $\phi_\alpha : U_\alpha \xrightarrow{\cong} (0,1)^m \subseteq \mathbb{H}^m$ (w/ std. orientation)

  (so $U_\alpha \cap \partial M = \emptyset$)

  OR

  (ii) $\phi_\alpha : U_\alpha \xrightarrow{\cong} (0,1] \times (0,1)^{m-1} \subseteq (\{x_1 \leq 1\}$ in $\mathbb{R}^m) \underset{\text{translate.}}{\cong} \mathbb{H}^m$. (w/ std. orientation)

  (so $U_\alpha \cap \partial M \underset{\phi_\alpha}{\cong} \{1\} \times (0,1)^m$).

Let $\{f_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$, and now let $\omega \in \Omega_c^{m-1}(M)$.

$$\int_M d\omega = \int_M \sum_\alpha \underbrace{d(f_\alpha \omega)}_{\text{supported in } U_\alpha} = \sum_\alpha \int_{U_\alpha} (d(f_\alpha \omega)) = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} \underbrace{(\phi_\alpha^{-1})^*(d(f_\alpha \omega))}_{= d(\underbrace{\phi_\alpha^{-1*}(f_\alpha \omega)}_{\omega_\alpha})} \Bigg\} \int_{\phi_\alpha(U_\alpha)} d\omega_\alpha$$

Each $\phi_\alpha(U_\alpha)$ is either $(0,1)^m$ or $(0,1] \times (0,1)^{m-1}$; by our verification of Stokes' thm in these cases, we get:

$$= \sum_\alpha \int_{\partial(\phi_\alpha(U_\alpha))} \omega_\alpha = \sum_\alpha \int_{\partial U_\alpha} f_\alpha \omega = \sum_\alpha \int_{\partial M} f_\alpha \omega = \int_{\partial M} \left(\sum_\alpha f_\alpha\right) \omega = \int_{\partial M} \omega.$$

$\boxed{\text{☒}}$

Integration on cohomology :

M oriented manifold (without ∂). As discussed Monday, $\int_M(-): \Omega^m_c(M) \to \mathbb{R}$ induces a non-trivial map $\int_M(-): H^m_c(M) \to \mathbb{R}$. (b/c $\int_M(\text{exact}) = 0$).

Observe: this map is always non-zero. why? Let ω be a nowhere vanishing top form on some $U_\alpha \xrightarrow[\text{oriented}]{\phi_\alpha} \phi_\alpha(U_\alpha) \subseteq \mathbb{R}^m$ representing the chosen orientation, & cutoff by a non-negative bump function to get $\psi\omega$ (extend by 0, so) $\psi\omega \in \Omega^m_c(M)$, and

$$\int_M \psi\omega = \int_{U_\alpha} \psi\omega = \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^*(\psi\,\omega) = \int_{\phi_\alpha(U_\alpha)} \tilde{\psi}\,\tilde{\omega} \;\boxed{> 0.}$$

$\tilde{\psi} = (\phi_\alpha^{-1})^*\psi$, $\tilde{\omega} = (\phi_\alpha^{-1})^*\omega$

$\tilde{\psi} \geq 0$ and >0 in some region.

$f\,dx_1 \wedge \cdots \wedge dx_m$ for $f > 0$

Hence $\int_M(-): H^m_c(M) \twoheadrightarrow \mathbb{R}$, so

$$\dim H^m_c(M) \geq 1 \text{ when } M \text{ is orientable.}$$

In fact, the following is true more generally

Thm: Say M^m is connected, orientable, without boundary.

Then, $\dim_{\mathbb{R}} H^m_c(M) = 1$

In fact: A choice of orientation on such M induces

$$\int_M(-): H^m_c(M) \xrightarrow{\cong} \mathbb{R}. \quad (*)$$

Cor: M compact, connected, orientable. Then

$\dim_{\mathbb{R}} H^m(M) = 1$ and given an orientation, get $\int_M(-): H^m(M) \xrightarrow{\cong} \mathbb{R}$.

Ex: S^n, we've seen that $H^k(S^n) = \begin{cases} \mathbb{R} & k=0,n \\ 0 & \text{otherwise.} \end{cases}$

Note: • For M non-compact, oriented, connected, $H^0(M) = \mathbb{R}$, & $H^0_c(M) = 0$ but $H^m_c(M) \cong \mathbb{R}$ (& it turns out $H^m(M) = 0$).

• For M cpct, oriented, connected, $H^0(M) \cong H^m(M) = \mathbb{R}$.

More generally, in fact the Poincaré Duality theorem (we won't prove in class, but you can read about it in e.g., Bott and Tu, Differential forms in Algebraic Topology):

there is a perfect pairing (given an orientation on M^m)

$$H^k(M) \times H^{m-k}_c(M) \xrightarrow{(-,-)_M} \mathbb{R}$$
$$[\alpha], [\beta] \longrightarrow \int_M \alpha \wedge \beta$$

Perfect means that $(-,-)_M$ induces an iso:

$$\left.\begin{aligned} H^k(M) &\xrightarrow{\cong} H^{m-k}_c(M)^* \\ H^k_c(M) &\xrightarrow{\cong} H^{m-k}(M)^* \end{aligned}\right] \text{ in particular } \dim H^k_c(M) = \dim H^{m-k}(M)$$

e.g., $\dim \underset{\underset{1 \text{ (if } M \text{ connected)}}{\parallel}}{H^0(M)} = \dim H^m_c(M)$,

(If M cpct, get $H^k(M) \cong H^{m-k}(M)^*$

$$\Rightarrow \dim H^k(M) = \dim H^{m-k}(M)$$

e.g., $\underset{\underset{1 \text{ (if } M \text{ connected)}}{\parallel}}{\dim H^0(M)} = \dim H^m(M)$)

Returning to today's thm: M^m connected, oriented. Then

$$\int_M(-) : H^m_c(M) \xrightarrow{\cong} \mathbb{R}. \quad (*)$$

Pf sketch: (i) Note we've already seen $\int_M(-)$ is surjective. So we just need injectivity. ($\Leftarrow$ if $\int_M \omega = 0$ then $[\omega] = 0$ i.e., then ω is exact.)

(ii) Note that if true for M, then for for any open $\mathcal{U}$, (e.g., $\mathcal{U} \xrightarrow{\cong} B_\varepsilon(p) \subset \mathbb{R}^m$), can find ω w/ $\operatorname{supp} \omega \subseteq \mathcal{U}$ & $\int_M \omega > 0$.

$\Rightarrow [\omega] \neq 0 \Rightarrow [\omega]$ spans $H^m_c(M)$.

So, if (*) holds, can find a representative of the generator of $H^m_c(M)$ w/ arbitrarily small support.

(iii) In general, injectivity of (*) amounts to showing that if $\int_M \omega = 0$ for $\omega \in \Omega^m_c(M)$, then $\omega = d\eta$, $\eta \in \Omega^{m-1}_c(M)$.

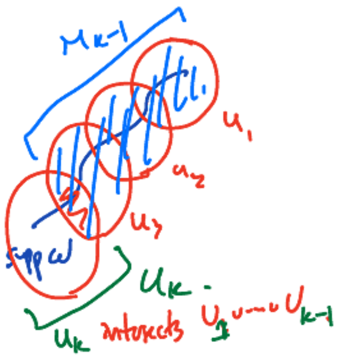
- Assume true for $M = \mathbb{R}^m$ $\boxed{\text{to check}}$ ☆, hence that it's true for general M if $\operatorname{supp}\omega \subseteq U \underset{\text{open}}{\subseteq} M$ with $U \cong \mathbb{R}^m$.

if $\omega \in M$ with $\operatorname{supp}(\omega) \subseteq U \subset M$, $U \cong \mathbb{R}^m$.
$\int_M \omega = 0 = \int_{\mathbb{R}^m} \omega = 0$
so ω is $d\eta$ w/ $\eta \in \Omega^{m-1}_c(\mathbb{R}^m)$
$\Rightarrow \omega = d\bar\eta$, $\bar\eta \in \Omega^{m-1}_c(M)$
extension of η by 0.

- General M, general ω: induct on # open sets $U_1, \dots, U_k$ required to cover $\operatorname{supp}\omega$ w/ $U_i \xrightarrow[\phi]{\cong} \mathbb{R}^m$ (# is finite by compactness of $\operatorname{supp}\omega$). By connectedness, assume $U_1 \cup \dots \cup U_{k-1}$ connected & intersects U_k.
  - true when $k = 1$.
  - Say true for $k-1$ open sets, & let $M_{k-1} = U_1 \cup \dots \cup U_{k-1}$.
    - Cut ω up into $\omega_{M_{k-1}} + \omega_{U_k}$ w/ (cpct) support in M_{k-1} & U_k resp., using a partition of unity adapted to $\{M_{k-1}, U_k\}$.
    - Each of $\omega_{M_{k-1}}$ & ω_{U_k} is cohomologous to forms θ_1, θ_2 supported in intersection $(U_k \cap M_{k-1}) \underset{\text{open}}{\subseteq} U_k \xrightarrow{\cong} \mathbb{R}^m$ by (ii). Hence

$$[\omega] = [\theta_1 + \theta_2], \text{ and } 0 = \int_M \omega = \int_{U_k} \theta_1 + \theta_2 = 0.$$

But $U_k \cong \mathbb{R}^m$, so $\theta_1 + \theta_2 = d\eta$ with $\eta \in \Omega^{m-1}_c(U_k) \subset \Omega^{m-1}_c(M)$.
Hence $[\theta_1 + \theta_2] = [\omega] = 0$. ∎



So by above, we've reduced the Thm to:

Lemma ☆: Fix std orientation on $\mathbb{R}^m$. Then $\int_{\mathbb{R}^m}(-) : H^m_c(\mathbb{R}^m) \xrightarrow{\cong} \mathbb{R}$.

Pf: (sketch) • Again, we know $\int_{\mathbb{R}^m}(-)$ is surjective by constructing a form w/ $\int > 0$.

- We've checked the lemma explicitly when $m = 1$ (example earlier in class). computed $H^1_c(\mathbb{R}^1)$ in an .
  (& true when $m = 0$ as well; $\int_{\mathbb{R}^0}(c) = c$).

- Inductively, one can show $\exists$ a map

$$\begin{array}{ccc} H^m_c(\mathbb{R}^m) & \xrightarrow[\cong]{P_*} & H^{m-1}_c(\mathbb{R}^{m-1}) \\ {\scriptstyle \int_{\mathbb{R}^m}(-)}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \int_{\mathbb{R}^{m-1}}(-)}\ \wr\!\wr \\ \mathbb{R} & = & \mathbb{R} \end{array} \quad (☆)$$

with P_* an isomorphism; this would complete the proof.

$\Rightarrow$ iso. by construction of G. iso. by induction.

$P: \Omega^k_c(\mathbb{R}^m) \to \Omega^{k-1}_c(\mathbb{R}^{m-1})$ linearly extends the following prescription:

$$f_I\, dx_m \wedge dx_I \longmapsto \Big(\underbrace{\int_{-\infty}^{\infty} f_I\, dx_m}\Big)\, dx_I$$

cpct support smooth fcn of $x_1 \to x_m$ — some indices not including m — function of $x_1 \to x_{m-1}$ now. — $\int_{-\infty}^{\infty} f_I(x_1, \ldots, x_{m-1}, t)\, dt$

$$g_J\, dx_J \longmapsto 0.$$

indices not including m

(exercise: P sends closed $\mapsto$ closed & exact $\mapsto$ exact, so induces cohomology map.

(exercise: P makes (#) commute.)

To show P is an isomorphism:

- Define $Q: \Omega^{k-1}_c(\mathbb{R}^{m-1}) \to \Omega^k_c(\mathbb{R}^m)$, by choosing $\chi \in C^\infty_c(\mathbb{R})$ with $\int_{\mathbb{R}} \chi(x)\,dx = 1$,

  and $\alpha \longmapsto \chi(x_m)\, \alpha \wedge dx_m$

  This induces $Q_*: H^{k-1}_c(\mathbb{R}^{m-1}) \to H^k_c(\mathbb{R}^m)$. (exercise)

- By construction $P \circ Q = \mathrm{id}_{\Omega^{k-1}_c(\mathbb{R}^{m-1})}$ so $P_* \circ Q_* = \mathrm{id}$ on cohomology.

- Define a chain homotopy K between $Q \circ P$ and id (maps from $\Omega^\bullet_c(\mathbb{R}^m) \to \Omega^\bullet_c(\mathbb{R}^m)$) given by

$$K_k: \Omega^k_c(\mathbb{R}^m) \to \Omega^{k-1}_c(\mathbb{R}^m)$$

$$f_I\, dx_m \wedge dx_I \longmapsto \Big(\int_{-\infty}^{\infty} f_I\, dx_m\Big)\Big(\int_{-\infty}^{x_m} \chi(t)\, dt\Big)\, dx_I$$

$$- \Big(\int_{-\infty}^{x_m} f(x_1, \ldots, x_{m-1}, t)\, dt\Big)\, dx_I.$$

$$g_I\, dx_I \longmapsto 0$$

terms not including dx_m.

check: $dK \pm Kd = Q \circ P \pm \mathrm{id}$, which implies $Q_* \circ P_* = \mathrm{id}$ on cohomology.

(c.f., Bott & Tu's book for more details).

(more generally, the same method proves $H^k_c(M \times \mathbb{R}) \xrightarrow{\cong} H^{k-1}_c(M)$ for any M.)

(in particular, we see how much $H^\bullet_c$ is not a homotopy invariant)

Extra material:

Q: How to see e.g., the divergence theorem as a special case of Stokes' theorem?

Recall

Thm: (Divergence thm): $\Omega \subset \mathbb{R}^3$ cpt. domain w/ smooth ∂, $\vec{F} = (F_1, F_2, F_3)$ a vector field on Ω. Then

$$\int_\Omega \operatorname{div} F \, dx\, dy\, dz = \int_{\partial\Omega} \langle n, F \rangle \, dA$$

where $n = (n_1, n_2, n_3)$

$$= n_1 \frac{\partial}{\partial x} + n_2 \frac{\partial}{\partial y} + n_3 \frac{\partial}{\partial z}$$ is the unit outward normal vector field to $\partial\Omega$,

and $dA = n_1\, dy \wedge dz + n_2\, dz \wedge dx + n_3\, dx \wedge dy$

$$= \iota_n (dx \wedge dy \wedge dz)$$

contraction along n.

(why is this dA? dA is defined so that for an oriented ONB of $\underset{T_p\mathbb{R}^3}{\overset{\shortparallel}{\mathbb{R}^3}}$ n_p, v_1, v_2 (normal at p), with v_1, v_2 tangent to $\partial\Omega$ at p $(v_1, v_2 \in T_p \partial\Omega)$, $dA(v_1, v_2) = 1$).

Pf of divergence thm (from Stokes' theorem):

Let $\omega = F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy. = \iota_{\vec{F}}\, dx \wedge dy \wedge dz.$

Then,

$$\int_\Omega d\omega = \int_\Omega (\operatorname{div} F)\, dx \wedge dy \wedge dz = \int_\Omega (\operatorname{div} F)\, dx\, dy\, dz.$$

std. orientation from $\mathbb{R}^3$

Stokes'

$\int \omega|_{\partial\Omega}$ Now claim: $\omega|_{\partial\Omega} = \langle n, F \rangle dA$, hence $\int_{\partial\Omega} \omega = \int \langle n, F \rangle dA$

$\partial\Omega$ $\quad {}_{\partial\Omega}$ $\quad {}_{\partial\Omega}$ $\quad {}_{\partial\Omega}$ $\quad {}_{\partial\Omega}$

(completing the proof).

Why? At a point $p \in \partial\Omega$, if $v_1, v_2 \in T_p\partial\Omega$,

then $\omega_p\big|_{T_p\partial\Omega}(v_1, v_2) = dx\wedge dy\wedge dz(\vec{F}, \vec{v}_1, \vec{v}_2)$

here we are thinking of elements of $\Lambda^k(T_p^*M)$ as elements of $\mathrm{AltMultilinear}(\underbrace{T_pM\times\cdots\times T_pM}_{k}, \mathbb{R})$ as in HW, where $k=2$.

$$= dx\wedge dy\wedge dz(\langle n, F\rangle n, v_1, v_2)$$

(b/c can write $\vec{F} = \langle n, F\rangle n + \vec{a}$ where $\vec{a} \in \mathrm{span}(v_1, v_2)$; by alternating multilinearity $dx\wedge dy\wedge dz(a, v_1, v_2) = 0$)

$$= \langle n, F\rangle (i_n\, dx\wedge dy\wedge dz)(v_1, v_2)$$

$$= \langle n, F\rangle\, dA(v_1, v_2), \text{ as desired.} \quad \blacksquare$$