Last time:

Stokes' thm: M manifold-with-boundary, oriented, $\omega \in \Omega_c^{m-1}(M)$.

Then $\int_M d\omega = \int_{\partial M} \omega$.

with ∂-induced orientation

- Showed true for $M = (0,1)^m$ or $(0,1] \times (0,1)^{m-1}$ directly (by extension to $[0,1]^m$ & FTC).
- (on virtual lecture notes-forthcoming): General proof follows by decomposing M into charts (U_α, ϕ_α) with $U_\alpha \xrightarrow[\phi_\alpha]{\cong}$ either $(0,1)^m$ or $(0,1]^m \times (0,1)^{m-1}$, (oriented)

if U_α in interior   if U_α intersects ∂.

& decomposing ω into sum of forms supported on each such chart using a partition of 1. $\Rightarrow$ reduces to above special cases.

Cor: M^m manifold, oriented (no ∂). Then $\int_M(-)$: exact cptly supported m-forms $\to 0$ $\left(\int_M d\omega = \int_{\partial M = \emptyset} \omega = 0\right)$

$$\Rightarrow \int_M(-) : H_c^m(M) \to \mathbb{R}.$$

Note: $\int_M(-)$ is surjective: $\exists$ "bump forms" supported in $U \subseteq M$ (with $U \cong \mathbb{R}^m$) with $\int_M \omega > 0$.

(automatically top forms are closed)

$\Rightarrow \dim H_c^m(M) \geq 1$.

Thm (from forthcoming lecture notes): If M is connected, oriented, (& no ∂) then

$$\int_M(-) : H_c^m(M) \xrightarrow{\cong} \mathbb{R}$$

(in particular, $\int_M(-)$ is injective too $\Leftrightarrow$ if $\int_M \omega = 0$ then ω is exact), so $[\omega] = 0$.

Note a consequence: If $\int_M \omega = \int_M \omega'$ then $[\omega] = [\omega']$.

In particular, any $[\omega]$ can be represented by a bump form σ supported in a given $U \subseteq M$ with $\int_M \sigma = \int_M \omega$.

The cohomological degree of a map (between cpct, oriented connected manifolds).

Now, say M^m is compact, (so $H^k_c(M) = H^k(M)$) oriented, connected.

The above theorem says that $\int_M (-) : H^m(M) \xrightarrow{\cong} \mathbb{R}$.

Say N^m is another compact, oriented, connected manifold of the same dimension m,

and let $\phi : M \longrightarrow N$ be a smooth map.

We can extract a number from ϕ as follows: by composing w/ integration as follows:

$$\begin{tikzcd}
H^m(M) \arrow[d, "\int_M"', "\cong"] & H^m(N) \arrow[l, "\phi^*"'] \arrow[d, "\int_N", "\cong"'] \\
\mathbb{R} & \mathbb{R} \arrow[l, dashed, "(*)", "\int_M(-)\circ\phi^*\circ\int_N(-)^{-1}"']
\end{tikzcd}$$

we get a linear map $\mathbb{R} \longrightarrow \mathbb{R}$, which must be multiplication by some scalar c_ϕ.

<u>Def</u>: The <u>cohomological degree</u> of $\phi : M^m \longrightarrow N^m$ is the unique scalar c_ϕ such that

cpct, connected, oriented, same dim.

for any $[\omega] \in H^m(N)$,

$$\int_M \phi^* \omega = c_\phi \int_N \omega .$$

(i.e., mult. by c_ϕ in $(*)$ makes the diagram commute).

<u>Prop</u>: c_ϕ only depends on the smooth homotopy class of ϕ; i.e., if $\phi \simeq \phi'$ then $c_\phi = c_{\phi'}$.

<u>Pf</u>: By homotopy invariance, $\phi^* = (\phi')^*$. ∎

There's another, more differential topological way to extract a number from ϕ:

<u>The (topological) degree of a map</u>:

Let $\phi : M^m \longrightarrow N^m$ smooth map with M, N cpct, oriented, connected.

- Pick a regular value $y \in N$; (exist by Sard's theorem).

Lemma: $\phi^{-1}(y)$ is finite.

Pf: Assume otherwise (so $\phi^{-1}(y)$ is infinite). Then, $\exists$ an accumulation point x (by compactness of M) (i.e., a limit point of a convergent subsequence), meaning any nhood of x contains infinitely many $p \in \phi^{-1}(y)$.

$\Rightarrow$ By continuity, $x \in \phi^{-1}(y)$.

Now, for every $p \in \phi^{-1}(y)$, $d\phi_p : T_pM \to T_yN$ is an isomorphism, in particular $d\phi_x : T_xM \to T_yN$ isomorphism.

$\underset{\text{(IFT)}}{\Rightarrow} \exists\, \mathcal{U} \ni x$ s.t. $\phi|_{\mathcal{U}}$ is a diffeomorphism $\Rightarrow$ $\mathcal{U} \not\ni q$ for any other $q \in \phi^{-1}(y)$, which is a contradiction as x was an accumulation point. $\boxtimes$.

Given a point $p \in M$, the $\boxed{\text{local degree of } \phi \text{ at } p}$ is

$$\boxed{\deg_p \phi = \begin{cases} +1 & d\phi_p : T_pM \to T_{\phi(p)}N \text{ orientation preserving} \\ -1 & \text{orientation reversing} \end{cases}}$$

Def: $\phi: M \to N$ as above. Define the $\boxed{\text{topological degree of } \phi}$ as follows:

- pick a regular value $y \in N$

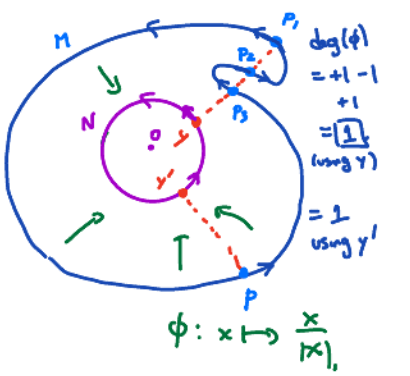
$$\rightsquigarrow \boxed{\deg(\phi) := \sum_{p \in \phi^{-1}(y)} \deg_p \phi \in \mathbb{Z}} \quad \text{(integer)}.$$

(finite sum by Lemma)

A priori, this depends on a choice of regular value but

Thm: This number is well-defined — independent of choice of regular value $y \in N$ of ϕ — and homotopic ϕ_0, ϕ_1 have the same degree.

Ex: $\deg(\phi) = 1$ computed using either y or y'



This Thm is an immediate consequence of

Thm: (topological = cohomological degree)

$$\boxed{\deg \phi = c_\phi}$$

topological degree (defined using any fixed regular value y) — cohomological degree

Note: It also follows that $c_\phi \in \mathbb{Z}$; a priori, we only knew $c_\phi \in \mathbb{R}$!

Pf: c_ϕ is defined to be the unique number s.t. $\int_M \phi^* \omega = c_\phi \int_N \omega$ for any ω, equivalently for some ω with $\int_N \omega \neq 0$.

(b/c such $[\omega]$ spans $H^m(N)$ & $\int$ only depends on $[\omega]$ & is linear).

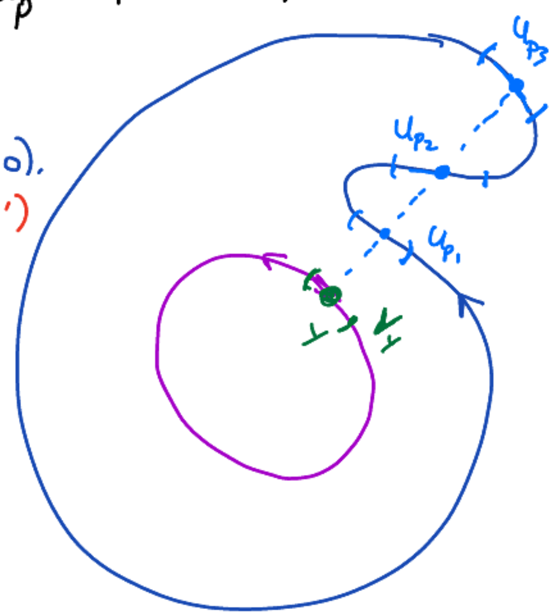
Let's pick a very specific ω (adapted to choice of regular value y) with $\int_N \omega \neq 0$ & show in fact that

$$\int_M \phi^* \omega = \deg(\phi) \int_N \omega$$

topological degree!

Let y be the regular value used to define $\deg(\phi)$. Pick a nbhd $V_y \ni y$ such that for every $p \in \phi^{-1}(y)$ $\exists$ $U_p \ni p$ such that $\phi|_{U_p} : U_p \xrightarrow{\cong} V_y$.

s.t., $V_y \cong \mathbb{R}^m$ oriented diff.



Now, pick $\omega \in \Omega^m(N)$ supported on V_y. (with $\int_N \omega \neq 0$). (e.g., 'bump form')

$\Rightarrow$ $\phi^* \omega$ is supported on $\coprod_{p \in \phi^{-1}(y)} U_p$.

so $\int_M \phi^* \omega = \sum_{p \in \phi^{-1}(y)} \int_{U_p} (\phi|_{U_p})^* \omega$

But each $U_p \xrightarrow[\phi|_{U_p}]{\cong} V_y$ diffeo, so

$$\int_{U_p} (\phi|_{U_p})^* \omega = \begin{cases} (+1) \cdot \int_{V_y} \omega & \text{if } \phi|_{U_p} \text{ orientation preserving} \Leftrightarrow d\phi_p \text{ orientation preserving} \\ (-1) \cdot \int_{V_y} \omega & \text{if } \phi|_{U_p} \text{ orientation reversing} \Leftrightarrow d\phi_p \text{ orientation reversing} \end{cases}$$

$$\Rightarrow \int_M \phi^* \omega = \Big(\sum_{p \in \phi^{-1}(y)} \deg_p \phi \Big) \underbrace{\int_{V_y} \omega}_{= \int_N \omega} = \deg(\phi) \int_N \omega. \quad \boxtimes$$

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ex: $S^1 \xrightarrow[z^2]{f_2} S^1$, $S^1 \subset \mathbb{C}$, $S^1 \subset \mathbb{C}$

check $\deg(f_2) = +2$.

<u>Some degree calculations & corollaries</u>:

$f : M^m \to N^m$ map between cpct, oriented, connected manifolds.

(1) If f diffeomorphism, then $\deg(f) = \pm 1$

(b/c $f^{-1}(y) = \{p\}$ one point, whose local degree can be only ± 1.
any point is a regular value

Note: $\boxed{\deg(id_M) = 1.}$

(2) If f is not surjective, then $\deg(f) = 0$.

why? Let $y \in N \setminus f(M)$ a point not in image(f). So $f^{-1}(y) = \emptyset$, so y a regular value

$\Rightarrow \deg(f)$ (computed using y) $= \sum_{p \in f^{-1}(y)} \deg_p(f) = 0$, b/c $\nexists\, p \in f^{-1}(y)$.

Cor: if f not surjective, y any reg. value, then $\#|f^{-1}(y)|$ is even.

$\Rightarrow$ If f is a constant map, $\deg(f) = 0$ (unless dimension$(M) = 0$).

Cor: If $n > 0$, then id_M is not smoothly homotopic to any constant map $c_p : M \to M$.

(3) From HW: if $A : S^n \to S^n$ is the antipodal map then

$$\deg(A) = \begin{cases} (-1) & n \text{ even} \\ 1 & n \text{ odd.} \end{cases}$$

$\Rightarrow$ A & id_{S^n} are not homotopic if n even.

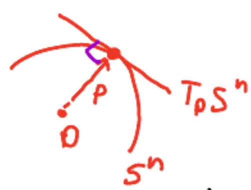
Cor: ("Hairy Ball theorem") Even dimensional spheres don't admit nowhere vanishing vector fields.

Pf: Let $S^n \subseteq \mathbb{R}^{n+1}$ unit sphere & X a non-vanishing vec. field on S^n.

By scaling assume $\|X_p\| = 1$ with respect to Euclidean inner product on $T_p S^n \subset T_p \mathbb{R}^{n+1} = \mathbb{R}^{n+1}$.

Recall $T_p S^n = \{ v \in T_p \mathbb{R}^{n+1} = \mathbb{R}^{n+1} \mid v \perp p \}$.

so $X_p \in T_p S^n$ means $X_p \perp p$.



Define $H(p,t) = \vec{p}\cos(\pi t) + \vec{X_p}\sin(\pi t)$ (move along great circle in direction of X_p at time t).

(note since $\vec{p} \perp \vec{X_p}$, $H(p,t) \in S^n \ \forall\, p, t$)

$H : S^n \times I \to S^n$.

H(p,1)

$H(p,0) = p$, i.e., $H(-,0) = id$

$H(p,1) = -p = A(p)$ $\quad H(-,1) = A$.

$\Rightarrow$ if X exists, then $id \simeq$ antipodal map $\Rightarrow$ n cannot be even. $\blacksquare$

$\deg(id) = 1$

$\deg(A) = (-1)^{n+1}$