Math 641 Homework 1: The Universal Coefficient and Künneth theorems

Due Wednesday Feb 15th, 2023 by 5 pm on Gradescope.

Please remember to write down your name and ID number. We will refer to pages/sections from Hatcher's *Algebraic Topology* by [Hatcher] and pages/sections from Bredon's *Topology and Geometry* by [Bredon]. You will only be graded on your solution to one of these problems. Nevertheless, you are encouraged to try to solve as many of these problems as you can. *If you are submitting a solution set consisting of more than one problem, please indicate clearly on your solution set which problem you'd like to be graded on.*

1. *Degree 1 cohomology classes.* Solve [Hatcher] §3.1 (page 205), problem 5.

2. *Verifying UCT in an example.* As discussed in class, the tools developed for homology translate directly into the cohomological setting, up to reversing directions of arrows. For example, when X is a CW complex, excision and long exact sequences imply there is a small **cellular co-chain complex** $(C^*_{CW}(X,G), \delta_{CW})$ computing the cohomology of X with G coefficients, and the Universal Coefficient Theorem implies that the complex is the G-dual of the cellular chain complex,

$$C^i_{CW}(X,G) = \mathrm{Hom}(C_i^{CW}(X), G). \tag{0.1}$$

   (See [Hatcher] §3.1, Theorem 3.5 on page 203 for a short proof spelling out the details).

   a. Compute $\mathrm{Ext}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ and $\mathrm{Tor}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ for natural numbers m, n directly from definitions. Include the degenerate case m or $n = 0$, where $\mathbb{Z}/0\mathbb{Z} = \mathbb{Z}$.
   b. Calculate $H^n(\mathbb{R}P^7, \mathbb{Z}/6\mathbb{Z})$ and $H_n(\mathbb{R}P^7; \mathbb{Z}/6\mathbb{Z})$, (i) starting from your knowledge of $H_n(\mathbb{R}P^7; \mathbb{Z})$ and the result in part (a), and then (ii) more directly from the cellular cochain complex.

3. *Non-naturality of the splitting in Universal Coefficient Theorem.* Solve [Hatcher] §3.1 (page 205), problem 11.

4. *Applications of UCT.*
   a. Show that $H^1(X)$ is always torsion free, for any X.
   b. Prove the following fact stated in class: singular homology groups of X are finitely generated, then the singular cohomology of X (with integral coefficients) admits the following description:

$$H^n(X) \cong F_n(X) \oplus T_{n-1}(X). \tag{0.2}$$

   Here $F_n(X)$ denotes the free subgroup of $H_n(X)$ and $T_{n-1}(X)$ denote the torsion subgroup of $H_{n-1}(X)$ (both with integral coefficients).

5. Let V and W be vector spaces over Λ. Show that the multiplication map $\hom(V, \Lambda) \times \hom(W, \Lambda) \to \hom(V \times W, \Lambda)$ need not be an isomorphism if both V and W are infinite dimensional. Specifically, show the map is not an isomorphism for $V = W = \Lambda^\infty = \oplus_{i=1}^\infty \Lambda$.

Conclude the Künneth isomorphism in cohomology fails in the case of $X = Y = \coprod_{i=1}^{\infty} \{\text{pt}\}$. *Hint*: If Q^* denotes $\hom(Q, \Lambda)$, note that the latter group is the same as $\hom(V, W^*)$, whereas the former is $V^* \otimes W^*$, and the multiplication map is the canonical map from $V^* \otimes Z$ to $\hom(V, Z)$ where $Z = W^*$.

6. *Computations using Künneth 1*: Solve [Bredon] p. 321 problem 2.

7. *Computations using Künneth 2*: Solve [Bredon] p. 321 problem 3.

8. *Associativity of cross product.* Solve [Bredon] p. 326 problem 1.