Math 641 Homework 2: The cup and cap product, orientations, Poincaré duality

Due Friday Feb 26, 2021 by 5 pm

Please remember to write down your name and ID number. We will refer to pages/sections from Hatcher's *Algebraic Topology* by [Hatcher] and pages/sections from Bredon's *Topology and Geometry* by [Bredon].

1. Show that if X can be covered by n acyclic open sets, then the cup product of any n cohomology classes of positive degree must be zero (This is [Hatcher] §3.2 (page 228), problem 2 or [Bredon] p. 334 problem 1, but note that the case of $n = 2$ is proved in [Bredon] Theorem 4.9; you can use this proof as a guide or a building block for how to case of general n).

2. *Computing the cohomology ring of a genus g surface.* Solve [Hatcher] §3.2 (page 228), problem 1.

3. *A computation using cup product.* Solve [Hatcher] §3.2 (page 229), problem 6.

4. *Distinguishing spaces using cup product.* Solve [Hatcher] §3.2 (page 229), problem 7.

5. *Cohomology rings with coefficients.* Solve [Hatcher] §3.2 (page 229), problem 9.

6. *The simplicial cup product.* Write down your favorite simplicial or Δ-complex structure on $\mathbb{R}P^2$ and use it to compute, via simplicial cohomology, the cohomology ring $H^*(\mathbb{R}P^2; \mathbb{Z}_2) \cong \mathbb{Z}_2[h]/h^3$ where $|h| = 1$.

7. *Orientability is unaffected by removing points.* Solve §3.3 (page 257), problem 2.

8. *The degree of maps between manifolds.* For a map $f : M \to N$ between connected closed orientable n-manifolds with fundamental classes $[M]$ and $[N]$, the **degree** of f is defined to be the integer d such that $f_*([M]) = d[N]$, so the sign of the degree depends on the choice of fundamental classes.

   a. *A map to S^n of degree 1.* Solve §3.3 (page 258), problem 7.

   b. *The degree of a covering map.* Solve §3.3 (page 258), problem 9.

   c. *The effect of degree 1 maps on π_1.* Solve §3.3 (page 258), problem 10.

9. *The homology groups of 3-manifolds.* Solve the first part of §3.3 (page 259), problem 24. Namely: let M be a closed, connected 3-manifold, and write $H_1(M; R)$ as $\mathbb{Z}^r \oplus T$, the direct sum of a free abelian group of rank r and a finite group T. Show that $H_2(M; \mathbb{Z})$ is $\mathbb{Z}^r$ if M is orientable and $\mathbb{Z}^{r-1} \oplus \mathbb{Z}/2\mathbb{Z}$ if M is non-orientable. In particular, $r \geq 1$ when M is nonorientable.

10. Show that if M is an odd-dimensional compact (not necessarily orientable) manifold, then $\chi(M) = 0$ (hint: show that $\chi(M)$ can be computed by homology with $\mathbb{Z}/2$ coefficients).