Math 641 Homework 2: The cup and cap product, orientations, Poincaré duality, vector bundles (double weight: please attempt two problems)

Due Friday April 7, 2023 by 5 pm

We will refer to pages/sections from Hatcher's *Algebraic Topology* by [HatcherAT], pages/sections from Bredon's *Topology and Geometry* by [Bredon], and pages/sections from Hatcher's *Vector bundles and K-theory* by [HatcherVB].

- 1. Show that if X can be covered by n acyclic open sets, then the cup product of any n cohomology classes of positive degree must be zero (This is [Hatcher] §3.2 (page 228), problem 2 or [Bredon] p. 334 problem 1, but note that the case of n = 2 is proved in [Bredon] Theorem 4.9; you can use this proof as a guide or a building block for how to case of general n).
- 2. Computing the cohomology ring of a genus g surface. Solve [Hatcher] §3.2 (page 228), problem 1.
- 3. A computation using cup product. Solve [Hatcher] §3.2 (page 229), problem 6.
- 4. Distinguishing spaces using cup product. Solve [Hatcher] §3.2 (page 229), problem 7.
- 5. Cohomology rings with coefficients. Solve [Hatcher] §3.2 (page 229), problem 9.
- 6. The simplicial cup product. Write down your favorite simplicial or Δ -complex structure on $\mathbb{R}P^2$ and use it to compute, via simplicial cohomology, the cohomology ring $H^*(\mathbb{R}P^2;\mathbb{Z}_2) \cong \mathbb{Z}_2[h]/h^3$ where |h| = 1. (recall that the Alexander-Whitney model of the cup product directly gives a combinatorial formula for the cup product on the simplicial co-chain complex).
- 7. Orientability is unaffected by removing points. Solve §3.3 (page 257), problem 2.
- 8. The degree of maps between manifolds. For a map $f: M \to N$ between connected closed orientable *n*-manifolds with fundamental classes [M] and [N], the **degree** of f is defined to be the integer d such that $f_*([M]) = d[N]$, so the sign of the degree depends on the choice of fundamental classes.
 - a. A map to S^n of degree 1. Solve §3.3 (page 258), problem 7.
 - b. The degree of a covering map. Solve §3.3 (page 258), problem 9.
 - c. The effect of degree 1 maps on π_1 . Solve §3.3 (page 258), problem 10.
- 9. The homology groups of 3-manifolds. Solve the first part of §3.3 (page 259), problem 24. Namely: let M be a closed, connected 3-manifold, and write $H_1(M; R)$ as $\mathbb{Z}^r \oplus T$, the

direct sum of a free abelian group of rank r and a finite group T. Show that $H_2(M;\mathbb{Z})$ is \mathbb{Z}^r if M is orientable and $\mathbb{Z}^{r-1} \oplus \mathbb{Z}/2\mathbb{Z}$ if M is non-orientable. In particular, $r \geq 1$ when M is nonorientable.

- 10. Show that if M is an odd-dimensional compact (not necessarily orientable) manifold, then $\chi(M) = 0$ (hint: show that $\chi(M)$ can be computed by homology with $\mathbb{Z}/2$ coefficients).
- 11. Let (E, π_E) and (F, π_F) be vector bundles over a common base M. A vector bundle morphism over M from E to F is a continuous map $f: E \to F$, compatible with projection, meaning that $\pi_F \circ f = \pi_E$, so that on each fiber $f_p: E_p \to F_p$ is a linear map of vector spaces. An isomorphism of vector bundles $E \cong F$ over potentially different spaces M and N is a homeomorphism $\overline{f}: M \to N$ and a vector bundle morphism covering \overline{f} which is an isomorphism on each fiber.¹ We say E and F are isomorphic over M if the isomorphism of vector bundles covers the identity map; e.g, if the vector bundle morphism $f: E \to F$ satisfies $\pi_F \circ f = \pi_E$, so f maps E_p to F_p .²

Let $(E, \pi : E \to M)$ be a rank k vector bundle. We say a collection of sections $s_1, \ldots, s_k \in \Gamma(E)$ is *linearly independent* if $(s_1)(p), \ldots, (s_k)(p)$ are linearly independent in E_p for each $p \in M$ (in particular, no $s_i(p)$ should be zero). Prove that there is an isomorphism of vector bundles over $M, E \cong \mathbb{R}^k$ if and only if E has a basis of linearly independent sections s_1, \ldots, s_k . (**Recall**: \mathbb{R}^k denotes the trivial rank k bundle over M, defined as $\mathbb{R}^k := M \times \mathbb{R}^k$, with projection map $\pi : M \times \mathbb{R}^k \to M$ sending (p, v) to p).

- 12. (a) Let $E = [0,1] \times \mathbb{R}/(0,t) \sim (1,-t)$ be the Möbius line bundle defined in class, with $\pi: E \to S^1 = [0,1]/0 \sim 1$ sending $(x,t) \mapsto x$. Verify that E is indeed a line bundle, and prove that E is not isomorphic to the trivial line bundle.
 - (b) Let $L = \{(x, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} | v \in x\}, \pi : L \to \mathbb{RP}^n, (x, v) \mapsto x$ denote the line bundle introduced in class; we call this bundle the *tautological line bundle on* \mathbb{RP}^n . Verify that L is indeed a line bundle.
- 13. Solve problems 2, and 3 of Section 1.1 of [HatcherVB] (the Vector Bundles and K-theory book), p. 17 (this counts for one problem).
- 14. Show that S^n admits a nowhere vanishing vector field if and only if n is odd. (Hint: if n is odd construct the vector field explicitly, and if n is even, show that the existence of a vector field would imply that id and the antipodal map are homotopic; now use degree

¹If $\bar{f}: M \to N$ is a continuous map, and $E, \pi_E : E \to M, F : \pi_F : F \to N$ are vector bundles, a *(general) bundle morphism covering* \bar{f} is a continuous map $f: E \to F$ satisfying $\pi_F \circ f = \bar{f} \circ \pi_E$, so that the induced map on fibers $E_p \to F_{\bar{f}(p)}$ is a linear map.

²An implicit point is that isomorphism of bundles over M is an equivalence relation. In particular, you should check that if there is an isomorphism of vector bundles $f: E \xrightarrow{\sim} F$, then f is a homeomorphism with $f^{-1}: F \to E$ also an isomorphism of vector bundles. You do not need to prove this fact as part of your HW, but if would like to use this fact on the HW elsewhere, you should prove it as a Lemma.

³By $v \in x$, we mean that, if x = [w], then $v \in Span(w)$.

theory to deduce a contradiction). Deduce that the tangent bundle TS^n is not trivial for n even.

In contrast, show that the normal bundle to $S^n \subset \mathbb{R}^{n+1}$ is always trivial. Recall that the normal bundle of a submanifold $Q \subset M$ is the quotient bundle $TM|_Q/TQ$ or equivalently (up to isomorphism) using any metric on TM, the fiberwise orthogonal complement of TQ inside $TM|_Q$.