Math 641 Homework 3: Bundles

Due Monday April 5, 2021 by 5 pm

Please remember to write down your name and ID number. We will refer to pages/sections from Hatcher's *Algebraic Topology* by [HatcherAT], pages/sections from Bredon's *Topology and Geometry* by [Bredon], and pages/sections from Hatcher's *Vector bundles and K-theory* by [HatcherVB].

1. Let (E, π_E) and (F, π_F) be vector bundles over a common base M. A *vector bundle morphism over M* from E to F is a continuous map $f : E \to F$, compatible with projection, meaning that $\pi_F \circ f = \pi_E$, so that on each fiber $f_p : E_p \to F_p$ is a linear map of vector spaces. An *isomorphism of vector bundles* $E \cong F$ over potentially different spaces M and N is a homeomorphism $\bar{f} : M \to N$ and a vector bundle morphism covering $\bar{f}$ which is an isomorphism on each fiber.[1] We say E and F are *isomorphic over M* if the isomorphism of vector bundles covers the identity map; e.g, if the vector bundle morphism $f : E \to F$ satisfies $\pi_F \circ f = \pi_E$, so f maps E_p to F_p.[2]

   Let $(E, \pi : E \to M)$ be a rank k vector bundle. We say a collection of sections $s_1, \ldots, s_k \in \Gamma(E)$ is *linearly independent* if $(s_1)(p), \ldots, (s_k)(p)$ are linearly independent in E_p for each $p \in M$ (in particular, no $s_i(p)$ should be zero). Prove that there is an isomorphism of vector bundles over M, $E \cong \underline{\mathbb{R}}^k$ if and only if E has a basis of linearly independent sections $s_1, \ldots, s_k$. (**Recall**: $\underline{\mathbb{R}}^k$ denotes the trivial rank k bundle over M, defined as $\underline{\mathbb{R}}^k := M \times \mathbb{R}^k$, with projection map $\pi : M \times \mathbb{R}^k \to M$ sending (p, v) to p).

2. (double weight) **Vector bundles via gluing data**. Let M be a topological space and $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ an open cover of M. A set of *transition* (or *gluing*) *data* (of rank k) for the cover $\mathcal{U}$ is a collection of continuous maps

$$\Phi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R}) \text{ for all } \alpha, \beta \in I$$

   satisfying

   i. For any $\alpha, \beta \in I$, $\Phi_{\alpha\beta} \cdot \Phi_{\beta\alpha} = I$.[3]
   ii. (*cocycle condition*) for any $\alpha, \beta, \gamma \in I$, $\Phi_{\gamma\alpha} = \Phi_{\gamma\beta} \cdot \Phi_{\beta\alpha}$ as functions on $U_\alpha \cap U_\beta \cap U_\gamma$.

   (a) Show, as asserted in class, that, given a collection of transition data $\mathcal{T} := \{\Phi_{\alpha\beta}\}_{\alpha,\beta \in I}$ for the cover $\mathcal{U}$, there is a rank k vector bundle $\mathcal{E}_{\mathcal{U},\mathcal{T}}$ over M, formed by gluing together

[1] If $\bar{f} : M \to N$ is a continuous map, and $E, \pi_E : E \to M$, $F : \pi_F : F \to N$ are vector bundles, a *(general) bundle morphism covering $\bar{f}$* is a continuous map $f : E \to F$ satisfying $\pi_F \circ f = \bar{f} \circ \pi_E$, so that the induced map on fibers $E_p \to F_{\bar{f}(p)}$ is a linear map.

[2] An implicit point is that isomorphism of bundles over M is an equivalence relation. In particular, you should check that if there is an isomorphism of vector bundles $f : E \xrightarrow{\sim} F$, then f is a homeomorphism with $f^{-1} : F \to E$ also an isomorphism of vector bundles. You do not need to prove this fact as part of your HW, but if would like to use this fact on the HW elsewhere, you should prove it as a Lemma.

[3] Here and in ii., multiplication means pointwise multiplication of matrices, i.e., for $f, g : V \to GL(k, \mathbb{R})$, $(f \cdot g)(p) = f(p) \cdot g(p)$. Also, I means the constant function with value the identity matrix; $V \to GL(k, \mathbb{R})$, $p \mapsto I$.

trivial vector bundles $U_\alpha \times \mathbb{R}^k$ via the transition data maps:

$$\mathcal{E}_{\mathcal{U},\mathcal{T}} := \coprod_{U_\alpha \in \mathcal{U}} (U_\alpha \times \mathbb{R}^k)/\{(U_\alpha, p, v) \sim (U_\beta, p, \Phi_{\alpha\beta}(p)(v)) \text{ for all } \alpha, \beta, p \in U_\alpha \cap U_\beta\}.$$

with projection map to M defined as follows: first, define

$$M_{\mathcal{U}} = \coprod_{U_\alpha \in \mathcal{U}} U_\alpha/\{(U_\alpha, p) \sim (U_\beta, p) \text{ for all } \alpha, \beta, p \in U_\alpha \cap U_\beta\};$$

check that $M_{\mathcal{U}}$ is indeed a topological space, and show that there is a canonical homeomorphism

$$i_{\mathcal{U}} : M_{\mathcal{U}} \xrightarrow{\sim} M$$

sending $[(U_\alpha, p)]$ to p. Then one defines $\pi : \mathcal{E}_{\mathcal{U},\mathcal{T}} \to M$ as the composition of the map $\pi_{\mathcal{T}} : \mathcal{E}_{\mathcal{U},\mathcal{T}} \to M_{\mathcal{U}}$ with $i_{\mathcal{U}}$, where $\pi_{\mathcal{T}} : [(U_\alpha, p, v)] \mapsto [(U_\alpha, p)]$ (again, you'll need to check this is continuous).[4]

Note: part of this problem involves verifying that (a) of $M_{\mathcal{U}}$ as above is indeed homeomorphic canonically to M, and (b) $\mathcal{E}_{\mathcal{U},\mathcal{T}}$ has the structure of a space with π a continuous map, and finally that (c) $(\mathcal{E}_{\mathcal{U},\mathcal{T}}, \pi)$ satisfy the axioms of a vector bundle.

(b) Show that any vector bundle E on M is isomorphic, as vector bundles over M (in the sense of problem 1), to $\mathcal{E}_{\mathcal{U},\mathcal{T}}$ for some cover $\mathcal{U}$ and some transition data $\mathcal{T}$.

(c) Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover and $\mathcal{T} = \{\Phi_{\alpha\beta}\}$ a set of transition data of rank k. A *(Čech) 0-co-chain* on $\mathcal{U}$ is a collection of functions $\mathcal{F} := \{f_\alpha : U_\alpha \to GL(k, \mathbb{R})\}$. Given such a Čech 0-co-chain, define new transition data for the cover $\mathcal{U}$ by

$$\mathcal{F}\mathcal{T} := \{f_\alpha^{-1} \cdot \Phi_{\alpha\beta} \cdot f_\beta\}.$$

Show that $\mathcal{E}_{\mathcal{U},\mathcal{T}}$ and $\mathcal{E}_{\mathcal{U},\mathcal{F}\mathcal{T}}$ are isomorphic as vector bundles over M.

(d) Given an open cover of M, $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$, along with transition data $\mathcal{T}$ for $\mathcal{U}$, a *refinement* of the cover $\mathcal{U}$ is a new open cover $\mathcal{U}' = \{V_\kappa\}_{\kappa \in J}$ along with a map of index sets $f : J \to I$ such that for each $\kappa \in J$,

$$V_\kappa \subset U_{f(\kappa)}.$$

(**Example**: If $M = \mathbb{R}^n$, one can take $\mathcal{U} = \{\mathbb{R}^n\}$ and $\mathcal{U}' = \{B_\epsilon(p)\}_{p \in J = \mathbb{R}^n}$. Then $\mathcal{U}'$, equipped with the canonical map on index sets $f : J \to \{*\}$, is a refinement of $\mathcal{U}$).

Given a refinement of $\mathcal{U}$, call it $\mathcal{U}'$, the *induced transition data on the refinement* is

$$\mathcal{T}|_{\mathcal{U}'} := \{\Phi_{f(\kappa)f(\tau)}|_{V_\kappa \cap V_\tau}\}_{\kappa,\tau \in J}.$$

Show that there is an isomorphism of vector bundles over M, $\mathcal{E}_{\mathcal{U}',\mathcal{T}|_{\mathcal{U}'}} \cong \mathcal{E}_{\mathcal{U},\mathcal{T}}$.

[4]**Note**: In this entire problem, for clarity we're using the notation $\coprod_{V \in \mathcal{V}} V := \{(V, p) | V \in \mathcal{V}, p \in V\}$. So elements of disjoint unions should be thought of as tuples (V, p) where V is one of the sets we're taking the disjoint union of and p is an element of V.

(e) Let $Vect^k(M)$ denote the set of rank k vector bundles on M up to isomorphism. That is,

$$Vect^k(M) := \{(E, \pi) \text{ a vector bundle over } M\}/\sim,$$

where $E \sim F$ if E and F are isomorphic over M, in the terminology of problem 1. Show that there is a bijection of sets

$$Vect^k(M) \cong \{(\mathcal{U}, \mathcal{T}) \mid \mathcal{U} \text{ is any open cover of } M \text{ and } \mathcal{T} \text{ is any transition data of rank } k \text{ for } \mathcal{U}\}/\sim,$$

where $(\mathcal{U}_1, \mathcal{T}_1) \sim (\mathcal{U}_2, \mathcal{T}_2)$ if, after passing to a common refinement $\mathcal{U}'$ of $\mathcal{U}_1$ and $\mathcal{U}_2$, there is a Čech 0-co-chain $\mathcal{F}$ on $\mathcal{U}'$ such that

$$\mathcal{T}_1|_{\mathcal{U}'} = \mathcal{F}(\mathcal{T}_2|_{\mathcal{U}'}).$$

Hint: there is a natural map $\{(\mathcal{U}, \mathcal{T}) \text{ as above}\} \to Vect^k(M)$ sending $(\mathcal{U}, \mathcal{T})$ to $(\mathcal{E}_{\mathcal{U},\mathcal{T}}, \pi)$. Show that this descends to a well-defined map $\{(\mathcal{U}, \mathcal{T}) \text{ as above}\}/\sim \to Vect^k(M)$ which is in fact a bijection.

3. (a) Let $E = [0,1] \times \mathbb{R}/(0,t) \sim (1,-t)$ be the Möbius line bundle defined in class, with $\pi : E \to S^1 = [0,1]/0 \sim 1$ sending $(x,t) \mapsto x$. Verify that E is indeed a line bundle, and prove that E is not isomorphic to the trivial line bundle.

   (b) Let $L = \{(x,v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} | v \in x\}$,[5] $\pi : L \to \mathbb{R}P^n$, $(x,v) \mapsto x$ denote the line bunde introduced in class; we call this bundle the *tautological line bundle on* $\mathbb{R}P^n$. Verify that L is indeed a line bundle.

4. Solve problems 2, and 3 of Section 1.1 of [HatcherVB] (the Vector Bundles and K-theory book), p. 17 (this counts for one problem).

5. Show that S^n admits a nowhere vanishing vector field if and only if n is odd. (Hint: if n is odd construct the vector field explicitly, and if n is even, show that the existence of a vector field would imply that *id* and the antipodal map are homotopic; now use degree theory to deduce a contradiction). Deduce that the tangent bundle TS^n is *not trivial* for n even.

   In contrast, show that the normal bundle to $S^n \subset \mathbb{R}^{n+1}$ is always trivial. *Recall that the normal bundle of a submanifold $Q \subset M$ is the quotient bundle $TM|_Q/TQ$ or equivalently (up to isomorphism) using any metric on TM, the fiberwise orthogonal complement of TQ inside $TM|_Q$.*

[5] By $v \in x$, we mean that, if $x = [w]$, then $v \in Span(w)$.