

Math 641 Homework 4: Characteristic classes

Due Monday, April 12, 2021 by 5 pm

Please remember to write down your name and ID number. We will refer to pages/sections from Milnor-Stasheff's *Characteristic classes* by [MilnorStasheff], and sections in Hatcher's *Algebraic Topology* by [HatcherAT].

1. [MilnorStasheff 4-D] If M^n can be immersed into \mathbb{R}^{n+1} show that the i th Stiefel-Whitney class $w_i(M)$ is equal to the i -th self-cup product of $w_1(M)$; that is $w_i(M) = w_1(M)^i$. Then show that if $\mathbb{R}P^n$ can be immersed in \mathbb{R}^{n+1} then n must be of the form $2^r - 1$ or $2^r - 2$.
2. [MilnorStasheff 4-E] Show that the set Ω consisting of all unoriented cobordism classes of smooth closed n -manifolds can be made into an additive group under disjoint union. This cobordism group is finite by Thom's result (that M is unoriented cobordant to N iff all their Stiefel-Whitney numbers are equal) and is a module over $\mathbb{Z}/2\mathbb{Z}$ (how?). Using the manifolds $\mathbb{R}P^2 \times \mathbb{R}P^2$ and $\mathbb{R}P^4$, show that Ω_4 consists of at least 4 elements.
3. w_1 and orientability Define the k th exterior power of a (real) vector bundle $E \rightarrow X$ of rank n for each k , and prove that $\wedge^n E$ has a section i.e., is trivial if and only if E is orientable. (A vector bundle is orientable if the associated bundle $Frame(E) \times_{GL(n)} (GL(n)/(GL^+(n)))$ has a section). Then prove, using e.g., the splitting principle, that if E is rank n then $w_1(\wedge^n E) = w_1(E)$. Using these ingredients, conclude that $w_1(E) = 0$ if and only if E is orientable.
4. Relationship between c_i and w_i Let S^{2m+1} be the unit sphere in \mathbb{C}^{m+1} , and $S^1 \subset \mathbb{C}^*$ the unit circle of complex numbers, acting on S^{2m+1} by complex multiplication. There is a principal $S^1 = U(1)$ -bundle, $S^{2m+1} \rightarrow \mathbb{C}P^m$ obtained by quotienting by the S^1 action. Consider now $\mathbb{R}P^{2m} = S^{2m+1}/\{\pm 1\}$; the induced projection map $\mathbb{R}P^{2m} \rightarrow \mathbb{C}P^m$ can be thought of as an $S^1/\{\pm 1\} = \mathbb{R}P^1$ bundle. We formally allow $m = \infty$ by taking the union of such fibrations for increasing m , thinking of $\mathbb{C}^{m+1} \subset \mathbb{C}^{m+2} \subset \dots$
 - (a) Prove that the Leray-Hirsch theorem with $\mathbb{Z}/2$ -coefficients applies to the fibration $\mathbb{R}P^{2m} \xrightarrow{\pi} \mathbb{C}P^m$ (with fibers $\mathbb{R}P^1$). Conclude that if $\bar{h} \in H^2(\mathbb{C}P^m; \mathbb{Z}_2)$ is the mod-2 reduction of the canonical generator $h \in H^2(\mathbb{C}P^m; \mathbb{Z})$, and if $H^*(\mathbb{R}P^{2m}; \mathbb{Z}_2) \cong \mathbb{Z}_2[w]$ with $|w| = 1$, that $\pi^* \bar{h} = w$.
 - (b) Prove that $\mathbb{R}P^{2m} \rightarrow \mathbb{C}P^m$ (including the case $m = \infty$) is the real projectivization of the tautological complex line bundle $L_{taut} \rightarrow \mathbb{C}P^m$, thought of as a real rank 2 bundle. Conclude from the definition of Stiefel-Whitney classes that $w_1((L_{taut})_{\mathbb{R}}) = 0$ and $w_2((L_{taut})_{\mathbb{R}}) = \bar{h} = c_1(L_{taut}) \pmod{2}$.
 - (c) Using the previous problem, prove more generally that for any complex vector bundle E , the total Stiefel-Whitney class of E thought of as a real bundle is the mod 2 reduction of the total Chern class, so $w(E_{\mathbb{R}}) = c(E) \pmod{2}$. In particular, $w_{2k+1}(E_{\mathbb{R}}) = 0$ and $w_{2k}(E_{\mathbb{R}}) = c_k(E) \pmod{2}$. *Hint: Prove this for all complex line bundles first using*

the previous calculation and universality. Now appeal to the splitting principle.

5. Show that $\mathbb{C}P^4$ (thought of as an 8-dimensional real manifold) does not embed in \mathbb{R}^{11} (hint: use Pontryagin classes).
6. *Computations of cohomology using Leray-Hirsch, [HatcherAT] §4.D (p. 447) #2.* Consider the action of $\mathbb{Z}_p \subset S^1$ on $S^{2n+1} \subset \mathbb{C}^{2n}$ by multiplying by complex roots of unity; stabilizing (under the inclusions $S^{2n+1} \subset S^{2n+3}$) we obtain an action of \mathbb{Z}_p on S^∞ . Denote by $B\mathbb{Z}_p := S^\infty/\mathbb{Z}_p$ (this is the classifying space for principal $G = \mathbb{Z}_p$ -bundles, and also the *first Eilenberg-MacLane space* $B\mathbb{Z}_p = K(\mathbb{Z}_p, 1)$; note that $B\mathbb{Z}_2 = \mathbb{R}P^\infty$).

Quotienting by the residual action of $S^1/\mathbb{Z}_p \xrightarrow{z \mapsto z^p} S^1$ gives a fibration $S^\infty/\mathbb{Z}_p \rightarrow \mathbb{C}P^\infty$ with fiber S^1 . Use this fibration and the Leray-Hirsch theorem to compute $H^*(B\mathbb{Z}_p; \mathbb{Z}_p)$ from $H^*(\mathbb{C}P^\infty; \mathbb{Z}_p)$.