Math 641, Spring 2023 - "Topics in topology" (this semester advanced algebraic topology, continuation of Math 540).

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Schedule: Most weeks, MW 9:30am–11am, backup time: F 9:30am – 11am

after that: MW, occasional F (makeup times).

Grading: • 50% HW assignments
- assigned every week or two, mostly optional
- each assignment, choose 1 problem to submit for a grade (& grade is for completion/effort).

• 50% final paper. (5-10 pages about a topic from a list of choices or another topic w/ instructor approval)

Overview of course: This is a second semester course in algebraic topology. (following Math 540, and to some extent, Math 535a), covering:

(1) cohomology theory & its algebraic structures (ring structure, module over Steenrod alg., ...)

(2) Poincaré duality for manifolds

(3) vector bundles & their characteristic classes

Time permitting, we'll also say something about

(4) K theory

(5) spectral sequences.

Some motivating (flavors of) questions we'll come back to:

(i) when are two spaces homotopy equivalent? (& e.g., which spaces are $\simeq$ manifolds?).

(e.g., take $\mathbb{CP}^3$ & $S^2 \times S^4$.

have: $H_*(\mathbb{CP}^3) = H_*(S^2 \times S^4)$ $\forall$ $*$.

& similarly $H^*(\mathbb{CP}^3) = H^*(S^2 \times S^4)$ as groups, but not as rings.

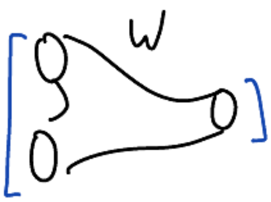
$\Rightarrow \mathbb{CP}^3 \not\simeq S^2 \times S^4$.

(ii) when are two maps $f_1, f_2 : X_1 \to X_2$ homotopic?

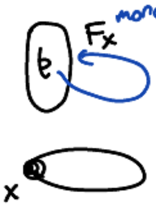
(iii) when are two vector bundles $\begin{matrix} E_1, E_2 \\ \searrow\swarrow \\ X \end{matrix}$ isomorphic?

(iv) When is there an embedding of manifolds $Z \hookrightarrow M$?

(v) when are M_1, M_2 cobordant? (cpct n-m'folds M_1, M_2 are cobordant if $\exists$ a cpct $(n+1)$-m'fld W^{n+1} (with ∂) s.t. $\partial W = M_1 \sqcup M_2$

e.g., $M_1 = S^1 \sqcup S^1$  $M_2 = S^1$.

(vi) Suppose we understand to build a space from simple spaces we understand. How to compute its invariants e.g., homology / cohomology?

(e.g., X = total space of fiber bundle $\begin{matrix} F \to X \\ \downarrow \pi \\ B \end{matrix}$, e.g., )

Definition of cohomology:

Recall singular homology, which is built out of singular chains

$$C_k(X) = \bigoplus_{\sigma : \Delta^k \to X} \mathbb{Z}\langle\sigma\rangle \quad \text{w/ } \partial_k : C_k(X) \to C_{k-1}(X)$$

"singular simplices"

$$\partial\sigma := \sum (-1)^i \sigma|_{[e_0, \ldots, \hat{e}_i, \ldots, e_k]}$$

"restriction of σ to $\partial\Delta^k = \coprod_{\text{faces}} \Delta^{k-1}$")

Had $\partial_{k-1} \circ \partial_k = 0$ (chain complex) and singular homology is def...

the $\partial_{(k-1)} \circ \partial_k = 0$ (chain complex), the singular homology is defined as:

$$H_k(X) := H_k\left(\{C_k(X), \partial_k\}_k\right) = \frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}}.$$

covariantly functorial: $f : X \to Y$ induces $f_\# : C_\bullet(X) \to C_\bullet(Y)$ &

$$f_* : H_\bullet(X) \to H_\bullet(Y).$$

& (local : excision/Mayer-Vietoris); • Have $H_\bullet(X,A) := H_k\left(C_\bullet(X,A) := C_\bullet(X) / C_\bullet(A)\right)$ (for $A \subset X$)

• $H_k(X;R)$ & $H_k(X,A;R)$ via $C_k(X;R) := C_k(X) \otimes_{\mathbb{Z}} R = \bigoplus_{\sigma : \Delta^k \to X} R\langle\sigma\rangle$.

We'll start by defining singular cohomology, a dual theory:

singular chains

$$C^k(X;R) := \operatorname{Hom}_{\mathbb{Z}}(C_k(X); R).$$

singular cochains

inherits $\delta_k = \partial_{k+1}^* : C^k(X;R) \to C^{k+1}(X;R)$

i.e., $\delta_{(k)}(f) := f \circ \partial_{(k+1)}$.

& similarly $\delta_{k+1} \circ \delta_k = 0$; so can define $H^k(X;R) := \dfrac{\ker \delta_k}{\operatorname{im} \delta_{k-1}}$.

(as before, have $H^k((X,A);R)$, Mayer-Vietoris/Excision, etc.,

contravariant functoriality ($f : X \to Y$ induces $f^* : H^\bullet(Y) \to H^\bullet(X)$).

Initially/a priori, this has same information as $H_k(X)$, packaged differently.

However, warning: ⚠ It is not always the case that $H^k(X;R) = \operatorname{Hom}_{\mathbb{Z}}(H_k(X);R)$.

That is: the operation of dualization does not 'commute' w/ that of taking (co)homology. The precise relationship between these two groups is known as the Universal coefficient theorem (UCT) for cohomology.