Universal coefficient theorems for homology and cohomology

Last time: X top. space $\underset{\text{last sem.}}{\rightsquigarrow}$ $C_\bullet(X)$ singular homology chain complex (w/ $\mathbb{Z}$-coefficients).

$$C_\bullet(X) = \{ \to C_n(X) \xrightarrow{\partial_{(n)}} C_{n-1}(X) \xrightarrow{\partial_{(n-1)}} \cdots \to C_0(X) \to 0 \}$$

boundary operators

chain complex: $\partial_{(n-1)} \circ \partial_{(n)} = 0$ $\quad\left(\rightsquigarrow \text{singular homology } H_n(X) := \dfrac{\ker \partial_n}{\operatorname{im} \partial_{n+1}} \right).$

From this, (rather than immediately take homology) we can form the singular ith co-chain group w/ G-coefficients (G any abelian group):

$$C^i(X;G) := \operatorname{Hom}_{\mathbb{Z}}(C_i(X), G) \qquad \text{(could take } G = \mathbb{Z}\text{, or something else)},$$

"Hom of Abelian groups = $\mathbb{Z}$-modules"

and the singular co-chain complex w/ G-coeffs.

$$C^\bullet(X;G) = \{ \cdots \xleftarrow{\delta} C^n(X;G) \xleftarrow{\delta} C^{n-1}(X;G) \leftarrow \cdots \xleftarrow{\delta} C^0(X;G) \leftarrow 0 \}$$

$= \operatorname{Hom}_{\mathbb{Z}}(C_\bullet(X), G)$ where $\delta_i = \partial^*_{i+1} = (-) \circ \partial_{i+1} : C^i(X;G) \to C^{i+1}(X;G)$ for all i.

Again we have $\delta \circ \delta = 0$ (called a co-chain complex),

added b/c differential increases degree

and can take $H^i(X;G) := \dfrac{\ker \delta^i : C^i(X;G) \to C^{i+1}(X;G)}{\operatorname{im} \delta^{i-1} : C^{i-1}(X;G) \to C^i(X;G)}.$

Rmks:
- For a pair (X,A) define $C^\bullet(X,A;G)$ by taking $\operatorname{Hom}_{\mathbb{Z}}(C_\bullet(X,A); G)$.

(made last class)
- $C_i(X) := \bigoplus_{\sigma : \Delta^i \to X} \mathbb{Z}\langle\sigma\rangle$, i.e., $C_i(X) = \operatorname{Free}(\operatorname{Sing}^i(X))$

set of all singular simplices $\sigma : \Delta^i \to X$.

So
$$\begin{aligned} C^i(X;G) &= \operatorname{Hom}_{\mathbb{Z}}(C_i(X), G) \\ &= \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Free}(\operatorname{Sing}^i(X)), G) \\ &= \operatorname{Maps}_{set}(\operatorname{Sing}^i(X), G). \end{aligned}$$

e.g.,
$$\begin{aligned} C^0(X;G) &= \operatorname{Maps}_{set}(\operatorname{Sing}^0(X), G) \\ &= \operatorname{Maps}_{set}(X, G) = \text{functions from } X^{discrete} \to G. \end{aligned}$$

X as a set (forgotten all topology).

- contravariant (as opposed to covariant) functoriality:

Any map $f: X \to Y$ induces $f_\#: C_\bullet(X) \to C_\bullet(Y)$ chain map,

hence induces $f^\# = (f_\#)^* = (-)\circ f_\# : C^\bullet(Y;G) \to C^\bullet(X;G)$

a co-chain map (meaning again $f^\# \circ \delta_Y = \delta_X \circ f^\#$), hence a map

$f^* = [f^\#]: H^\bullet(Y;G) \to H^\bullet(X;G)$ (in contrast the same f induces $f_*: H_\bullet(X) \to H_\bullet(Y)$)

(similarly for $f:(X,A) \to (Y,B)$, get f^* between cohomologies in opposite direction)

In light of the fact that $C^\bullet(X;G)$ are determined as $\mathrm{Hom}_{\mathbb{Z}}(C_\bullet(X), G)$, we might ask:

Q: what's the relationship between $H^\bullet(X;G)$ and $H_\bullet(X)$?

To put things on level footing, let's recall we can also take singular chains w/ G-coeffs: (G any ab. group)

$C_n(X;G) := C_n(X) \otimes_{\mathbb{Z}} G$, w/ induced ∂ ($:= \partial \otimes id_G$), and

$\rightsquigarrow H_\bullet(X;G)$

implicitly $H_n(X;\mathbb{Z})$

Q: what's the relationship between $H_n(X)$ and $H_n(X;G)$?

More generally, can consider any chain complex $C_\bullet = \{ \to C_n \xrightarrow{\partial} \cdots \to C_0 \to 0\}$;

Q: what's the relation between $\{H_n(C_\bullet)\}_n$ and

- $\{H^n(\mathrm{Hom}_{\mathbb{Z}}(C_\bullet, G))\}_n$?
- $\{H_n(C_\bullet \otimes_{\mathbb{Z}} G)\}_n$?

Example:

deg 2 deg 1 deg 0

$$C = \quad 0 \to \mathbb{Z}\oplus\mathbb{Z} \xrightarrow[\partial_1]{[2,0]} \mathbb{Z} \longrightarrow 0$$ has homology

$H_1 = \ker \partial_1 \cong \mathbb{Z}$ ($\mathbb{Z}\oplus 0$)

$H_0 = \mathrm{coker}\, \partial_1 \cong \mathbb{Z}/2\mathbb{Z}$.

$$\mathrm{Hom}(C,\mathbb{Z}): \quad 0 \leftarrow \mathbb{Z}\oplus\mathbb{Z} \xleftarrow[\delta_0]{\begin{bmatrix}2\\0\end{bmatrix}} \mathbb{Z} \leftarrow 0$$

not linear duals of each other!!

$\operatorname{Hom}(\mathbb{Z}^2,\mathbb{Z}) \cong \mathbb{Z}^2$

$$H^1 = \operatorname{coker} \delta_0 = \frac{\mathbb{Z}\oplus\mathbb{Z}}{2\mathbb{Z}\oplus 0} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}.$$

$$H^0 = \ker \delta_0 = 0$$

Roughly speaking, it seems "free part of H_i contributes to H^i, torsion part of H_{i-1} contributes to H^i." (but so far, this is an imprecise idea).

$$C \otimes_{\mathbb{Z}} \mathbb{Z}/2 : \quad 0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{[0\ 0]} \mathbb{Z}/2 \to 0$$

$$\rightsquigarrow H_i(C_\bullet; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 & i=1 \\ \mathbb{Z}/2 & i=0 \end{cases} \quad (\text{not } H_i(C_\bullet) \otimes_{\mathbb{Z}} \mathbb{Z}/2 \,!)$$

Exercise: look at $\operatorname{Hom}(C, \mathbb{Z}/2)$ to see similar discrepancies

General $C_\bullet$:

Note there is a natural map

$$H^n = H^n(\operatorname{Hom}(C_\bullet, G)) \xrightarrow{\beta} \operatorname{Hom}_{\mathbb{Z}}(H_n(C_\bullet), G)$$

with $[f]$ in $H^n(\operatorname{Hom}(C_\bullet, G))$, given by

$$\beta([f])([c]) := f(c).$$

(exercise: this is independent of choice of representatives of $[f]$ and $[c]$, i.e., note that

$$(f+\delta g)(c) = f(c) + \delta g(c) = f(c) + g(\partial c) \quad (\text{b/c } \delta g = g \circ \partial)$$
$$= f(c) \quad (\text{b/c } \partial c = 0).)$$

There's also a natural map

$$H_n(C_\bullet) \otimes G \xrightarrow{\alpha} H_n(C_\bullet \otimes G)$$

$$\alpha([x] \otimes g) := [x \otimes g]$$

More refined question: how to measure failure of β, α to be isomorphisms?

note $C_\bullet(X)$ is free.

Theorem: (universal coefficient theorem for cohomology) (UCT) For any free chain complex $C_\bullet$ (means each C_i is free), there is a (functorial) natural in $C_\bullet$ and G SES for each n:

the map β from above

$$0 \to \operatorname{Ext}(H_{n-1}(C_\bullet), G) \to H^n(\operatorname{Hom}(C_\bullet, G)) \xrightarrow{\beta} \operatorname{Hom}_{\mathbb{Z}}(H_n(C_\bullet), G) \to 0$$

new term! Sometimes called $\mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}, G)$ (but we often leave $\mathbb{Z}$, 1 implicit; can leave 1 implicit b/c $\mathrm{Ext}^k_{\mathbb{Z}}(A,B) = 0$ for $k>1$).

Furthermore, this sequence <u>splits</u> (naturally in G, but <u>not</u> <u>naturally</u> in $C_\bullet$).

Recall: A SES $0 \to A \xrightarrow{i} B \underset{\exists k}{\overset{j}{\rightleftarrows}} C \to 0$ splits if $\exists\ k: C \to B$ w/ $j \circ k = \mathrm{id}_C$

If k exists, it need not be unique, and k induces $A \oplus C \underset{\cong}{\xrightarrow{(i,k)}} B$, so get $B \cong A \oplus C$.

non-split ex:

$$0 \to \mathbb{Z} \xrightarrow{i = \times 2} \mathbb{Z} \xrightarrow{j = \text{projection}} \mathbb{Z}/2 \to 0..$$

So UCT (+ a choice of splitting) gives

$$H^n(\mathrm{Hom}_{\mathbb{Z}}(C_\bullet, G)) \cong \mathrm{Hom}_{\mathbb{Z}}(H_n(C_\bullet), G) \oplus \mathrm{Ext}(H_{n-1}(C_\bullet), G).$$

but this isomorphism is not <u>natural</u> in $C_\bullet$; i.e., a chain map $f: C_\bullet \to C_\bullet'$ induces a map of SES's above (& in particular on cohomology), but not necessarily respecting the direct sum decompositions for any choice of splitting. (<u>exercise</u>)

<u>Thm</u>(UCT for homology): For a free chain complex $C_\bullet$, $\exists$ an exact sequence

$$0 \to H_n(C_\bullet) \otimes_{\mathbb{Z}} G \xrightarrow{\alpha} H_n(C_\bullet \otimes G) \to \mathrm{Tor}(H_{n-1}, G) \to 0$$

sometimes called $\mathrm{Tor}_1^{\mathbb{Z}}$ (but can suppress $\mathbb{Z}$ if implicit & 1 again b/c $\mathrm{Tor}^{\mathbb{Z}}_{>1} \equiv 0$).;

natural (functorial) in $C_\bullet$ (w/r/t. chain maps) and G. The sequence splits (naturally in G, but <u>not</u> in $C_\bullet$).

The next goal is to define Ext/Tor, then we'll see how to prove UCTs.

For any R-modules M, N, can define $\mathrm{Ext}^i_R(M,N)$ and $\mathrm{Tor}^R_i(M,N)$, with

commutative ring (only so we don't have to worry about left vs. right modules; R associative is otherwise sufficient).

$$\mathrm{Ext}^0_R(M,N) = \mathrm{Hom}_R(M,N), \quad \mathrm{Tor}^R_0(M,N) = M \otimes_R N;$$

with $\mathrm{Ext}^i_R(M,N)$ (resp. $\mathrm{Tor}^R_i(M,N)$) for $i>0$ measuring "the failure at M" of $\mathrm{Hom}_R(-,N)$ (resp. $(-)\otimes_R N$) to be 'exact.'"

A functor f is <u>exact</u> if whenever have SES $0 \to A \to B \to C \to 0$ then

- if f contravariant,

$0 \to f(C) \to f(B) \to f(A) \to 0$ is exact.

- if f covariant

$$0 \to f(A) \to f(B) \to f(C) \to 0 \text{ is exact.}$$

— case of $\mathrm{Hom}_{\mathbb{Z}}$:

Note: If $0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ is exact, then exercise:

ex: $0 \to \underset{(\mathbb{Z})}{\mathrm{Hom}}(A'', B) \xrightarrow{j^*} \mathrm{Hom}(A, B) \xrightarrow{i^*} \mathrm{Hom}(A', B)$ is exact, but

i^* need not be surjective, i.e., not every map $A' \to B$ is the restriction of a map $A \to B$, i.e., not every map $A' \to B$ extends to $A \to B$. (n.b. Ext stands for "extension").

counter-ex: $0 \to \underset{A'}{\mathbb{Z}} \xrightarrow{\times 2} \underset{A}{\mathbb{Z}} \to \mathbb{Z}/2 \to 0$ induces

$$\underset{\mathbb{Z}}{\overset{\shortparallel}{\mathrm{Hom}(A,B)}} \xrightarrow{\times 2} \underset{\mathbb{Z}}{\overset{\shortparallel}{\mathrm{Hom}(A',B)}}$$

, which is not surjective.

Rmk: If $0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ is split SES, then in fact get a SES

$$0 \to \mathrm{Hom}(A'', B) \xrightarrow{j^*} \mathrm{Hom}(A, B) \xrightarrow{i^*} \mathrm{Hom}(A', B) \to 0.$$

$\cong$ by splitting

$\mathrm{Hom}(A' \oplus A'', B)$

easy to see this map surjects ("extend by 0").

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Similarly, if $0 \to A' \xrightarrow{i} A \to A'' \to 0$ exact, then we are guaranteed only that

$$A' \otimes_{\mathbb{Z}} B \xrightarrow{i \otimes id_B} A \otimes_{\mathbb{Z}} B \to A'' \otimes_{\mathbb{Z}} B \to 0$$

is exact; $i \otimes id_B$ need not be injective.

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How to measure these "failures of exactness"?

Use projective (or injective) resolutions as a 'replacement' of our given group/module.

(particularly nice modules for which the above problems don't arise)

Def:

An R-module Q is an injective R-module if, for any injective map (of R-modules) $f: M \to N$,

and any map $g: M \to Q$ $\exists$

$$\text{(SES)} \quad \begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "g"] & N \arrow[dl, dashed] \\ & Q & \end{tikzcd}$$

$\exists\, h$ with $h \circ f = g$.

"any g to Q extends along injections".

(injective)

(injective module) Q

$\overset{\text{exercise:}}{\Longleftrightarrow}$ if any SES $0 \to Q \to M \to K \to 0$ splits.

$\overset{\text{exercise}}{\Longleftrightarrow}$ if $\mathrm{Hom}_R(-, Q)$ is exact.

Defs An R-module P is <u>projective</u> if for any surjection $f: N \to M$ and any map $g: P \to M$, $\exists$:

$$\begin{tikzcd} N \arrow[r, "f"] & M \arrow[r] & 0 \\ & P \arrow[u, "g"'] \arrow[ul, dashed, "\exists h"] & \end{tikzcd}$$

(SES)

with $f \circ h = g$.

(projective)

"any g from P lifts along surjections."

$\Longleftrightarrow$ any $\overset{\text{SES}}{0} \to A \to B \to P \to 0$ is split.

$\Longleftrightarrow$ $\mathrm{Hom}(P, -)$ is exact.

lecture end

Thm: (exercise or look up in a book): For a $\mathbb{Z}$-module M (i.e., an abelian group) (or more generally M over a <u>PID</u> R).

- M is injective iff it is <u>divisible</u>.

  (an abelian group G is divisible if for any $g \in G$ and any $n \in \mathbb{N}$, $g = n \cdot g'$ for some $g' \in G$)

  <u>ex</u>: $\mathbb{Q}$, <u>non-ex</u>: $\mathbb{Z}$, or $\mathbb{Z}/2$

  but $\mathbb{Z}/2$ is injective as a $\mathbb{Z}/2$-module!

- M is projective iff it is free.

<u>Cor</u>: For a projective $\mathbb{Z}$-module P, given any injection $0 \to P' \to P$ (i.e., $\hookrightarrow$ a submodule), P' is projective too. (subgroups of free abelian groups are free abelian)

Similarly, if Q injective $\mathbb{Z}$-module, $Q \to Q' \to 0$, Q' injective too.