Ext/Tor:

Goal: For R modules M, N, define

$$\mathrm{Ext}^i_R(M,N)$$
$$\mathrm{Tor}_j^R(M,N)$$

w/ $\mathrm{Ext}^0_R(M,N) = \mathrm{Hom}_R(M,N)$, $\mathrm{Tor}_0^R(M,N) = M \otimes_R N$,

- $\mathrm{Ext}^{>1}_{\mathbb{Z}}(M,N) = 0$ so use shorthand $\mathrm{Ext}(M,N)$ for $\mathrm{Ext}^1_{\mathbb{Z}}(M,N)$, similar for $\mathrm{Tor}^{\mathbb{Z}}$.

Last time:

- An R-module Q is an injective R-module if, for any injective map (of R-modules) $f: M \to N$, and any map $g: M \to Q$ $\exists$

(SES)
$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "g"] & N \arrow[dl, dashed] \\ & Q & \end{tikzcd}$$

$\exists\, h$ with $h \circ f = g$.

(injective module) Q

"any g to Q extends along injections".

$\overset{\text{exercise:}}{\iff}$ if any SES $0 \to Q \to M \to K \to 0$ splits.

(Rmk: $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ splits iff $B \cong A \oplus C$ w/ i inclusion, j projection

$\iff \exists\ 0 \to A \to B \underset{s}{\leftrightarrows} C \to 0$ w/ $j \circ s = \mathrm{id}_C \iff$

$\exists\ 0 \to A \underset{s'}{\leftrightarrows} B \to C \to 0$ w/ $s' \circ i = \mathrm{id}_A$)

$\overset{\text{exercise}}{\iff}$ if $\mathrm{Hom}_R(-, Q)$ is exact.

- An R-module P is projective if for any surjection $f: N \to M$ and any map $g: P \to M$, $\exists$:

$$\begin{tikzcd} N \arrow[r, "f"] & M \arrow[r] & 0 \\ & P \arrow[u, "g"] \arrow[ul, dashed, "\exists h"] & \end{tikzcd}$$

(SES)

"any g from P lifts along surjections."

with $f \circ h = g$ (projective)

$\iff$ any$^{\text{SES}}$ $0 \to A \to B \to P \to 0$ is split.

$\iff$ $\mathrm{Hom}(P, -)$ is exact.

<u>Thm</u>: (exercise or look up in a book): For a $\mathbb{Z}$-module M (i.e., an abelian grp) (or more generally M over a <u>PID</u> R).

- M is injective iff it is <u>divisible</u>.
  (an abelian grp G is divisible if for any $g \in G$ and any $n \in \mathbb{N}$, $g = n \cdot g'$ for some $g' \in G$)
  <u>ex</u>: $\mathbb{Q}$, <u>non-ex</u>: $\mathbb{Z}$, or $\mathbb{Z}/2$
  ↖ but $\mathbb{Z}/2$ is injective as a $\mathbb{Z}/2$-module!

- M is projective iff it is free. (for any R, free $\Rightarrow$ projective, not always $\Leftarrow$)

<u>Cor</u>: For a projective $\mathbb{Z}$-module P, given any injection $0 \to P' \to P$ (i.e., $\hookrightarrow$ a submodule), P' is projective too. (we subgroups of free abelian groups are free abelian)

Similarly, if Q injective $\mathbb{Z}$-module, $Q \to Q' \to 0$, Q' injective too.

A <u>projective resolution</u> of an R-module M is an <u>exact</u> sequence

$$\to \cdots \to P_2 \to P_1 \to M \to 0$$

(resp. <u>injective</u> resolution:

$$0 \to Q \to I_1 \to I_2 \to \cdots \cdots).$$

It turns out those resolutions always exist, and

as a consequence of Thm / Cor above, we deduce:

Any $\mathbb{Z}$-module (i.e., abelian grp) M has a two-step projective (resp. injective) resolution

$$0 \to P_1 \to P_0 \xrightarrow{f} M \to 0.$$ (why? Pick surjection $f: P_0 \to M$ & have $0 \to \ker f \to P_0 \to M \to 0$; $\ker f$ proj. by cor)

(resp. $0 \to M \to I_0 \to I_1 \to 0$).

<u>Def</u>: Given R-modules M, N, pick projective resolution of M

$$(\cdots \xrightarrow{f_1} P_1 \xrightarrow{f_0} P_0) \to M \to 0,$$

take $\mathrm{Hom}_R(-,N)$ of all P_i's & f_i's between them (but not of M):

$$- - - - \xleftarrow{f_1^*} \mathrm{Hom}(P_1,N) \xleftarrow{f_0^*} \mathrm{Hom}(P_0,N) \leftarrow 0$$

This is no longer exact; however it's a chain complex: $f_{i+1}^* \circ f_i^* = 0$. (so $\ker f_{i+1}^* \supset \mathrm{im} f_i^*$).

We define $\mathrm{Ext}^i_R(M,N) := \dfrac{\ker f_i^*}{\mathrm{im}\, f_{i-1}^*}$. Note $\mathrm{Ext}^0_R(M,N) = \ker f_0^* \cong \mathrm{Hom}_R(M,N)$

(b/c $P_1 \xrightarrow{f_0} P_0 \to M \to 0$ induces an exact sequence

$0 \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(P_0,N) \xrightarrow{f_1^*} \mathrm{Hom}(P_1,N)$)

This a priori depends on a choice (of proj. resolution); however

Thm: $\mathrm{Ext}^i_R(M,N)$ are independent of choices made and functorial in M/N.

Rmk: Can also define $\mathrm{Ext}^i_R(M,N)$ by injectively resolving N, exercise that the result is unchanged.

Cor: For $\mathbb{Z}$-modules M,N, $\mathrm{Ext}^i_{\mathbb{Z}}(M,N) = 0$ for $i > 1$.

(b/c M admits a two-term proj. resolution $0 \to P_1 \to P_0 \to M \to 0$).

Similarly, to define $\mathrm{Tor}_j^R(M,N)$, projectively resolve either M or N:

$\underbrace{(- - - \to P_1 \to P_0)}_{P_\bullet} \to M \to 0$, then take $P_\bullet \otimes_R N$; again the result is a chain complex (no longer exact); jth homology is $\mathrm{Tor}_j^R(M,N)$.

& same Thm: functorial + independent of choice.

Both these theorems follow from "homotopically unique functoriality of projective resolutions":

(*) Thm: Say $P_\bullet \to M$ is a proj. resolution and $\bar{f}: M \to N$ is a map (of R-modules), and $Q_\bullet \to N$ a proj. resolution. Then there is a chain map $f_\bullet : P_\bullet \to Q_\bullet$ 'lifting' $\bar{f}$,

meaning:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial^P"] \arrow[d, dotted] & P_1 \arrow[r, "\partial^P"] \arrow[d, dotted, "\exists f_2"] & P_0 \arrow[r] \arrow[d, dotted, "\exists f_1"] & M \arrow[r] \arrow[d, "\bar{f}"] & 0 \\
\cdots \arrow[r, "\partial^Q"'] & Q_1 \arrow[r, "\partial^Q"'] & Q_0 & N \arrow[r] & 0
\end{tikzcd}$$

making diagram commute.

and f is unique up to homotopy equivalence (meaning if f, f' two lifts then $\exists\ h: P_\bullet \to Q_{\bullet+1}$ with $f - f' = \partial^Q \circ h + h \circ \partial^P$).

— This induces, for $f: M \to N$, a map f^*: between co-chain complexes computing $\mathrm{Ext}^\bullet_R(N,S)$ & $\mathrm{Ext}^\bullet_R(M,S)$.

To see from this that e.g., Ext is indep of choices, let $N = M$, $\bar{f} = \mathrm{id}$, $P_\bullet$, $Q_\bullet$ two different resolutions of M.

(✻) $\Rightarrow$ get a chain map $f: P_\bullet \to Q_\bullet$ lifting id, & hence (by taking $\mathrm{Hom}_R(-,S)$) a chain map between Ext complexes, f^*

we also get a map $g: Q_\bullet \to P_\bullet$ & get g^* other way between Ext complexes:

By homotopical uniqueness, $f \circ g$ & $g \circ f$ are each homotopic to $\mathrm{id}_{Q_\bullet}$, $\mathrm{id}_{P_\bullet}$ respectively, hence f^*, g^* induce isomorphisms on Ext grps.

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Sketch of proof of theorem: idea is to inductively use the projectivity condition:

inductively say we've constructed $f_i, \dots, f_0$. (base case: $\bar{f}$ itself $i = -1$).

$$\begin{tikzcd}
\cdots \arrow[r, dashed] & P_{i+1} \arrow[r, "\partial^P"] \arrow[d, dashed, "\exists ?\ f_{i+1} ?"] & P_i \arrow[r, "\partial^P_i"] \arrow[d, "f_i"] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "f_0"] & M \arrow[d, "\bar{f}"] \\
\cdots \arrow[r, dashed] & Q_{i+1} \arrow[r, "\partial^Q"'] & Q_i \arrow[r, "\partial^Q"] & \cdots \arrow[r] & Q_0 \arrow[r] & N
\end{tikzcd}$$

want: $\partial^Q \circ f_{i+1} = f_i \circ \partial^P$. Since sequences at top and bottom are exact, we can restrict to:

$$\begin{tikzcd}
P_{i+1} \arrow[r] \arrow[d, dashed, "\exists \text{ by projectivity}"'] \arrow[dr, "\text{map from proj.}"] & \underbrace{\ker(\partial^P_i)}_{=\,\mathrm{im}(\partial^P_{i+1})} \arrow[r] \arrow[d, "f_i"] & 0 \\
Q_{i+1} \arrow[r, "\text{surjective}"'] & \ker(\partial^Q_i) \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & \ker(\partial^P_i) \arrow[r] \arrow[d, "f_i"] & P_i \arrow[r] \arrow[d, "f_i"] & \cdots \arrow[d, "f_{i-1}"] \\
0 \arrow[r] & \ker(\partial^Q_i) \arrow[r] & Q_i \arrow[r] & \cdots
\end{tikzcd}$$

Uniqueness up to homotopy? exercise. Similar inductive level-by-level construction of

$$\begin{tikzcd}
\cdots \arrow[r] & P_{i+1} \arrow[r] \arrow[d, "f_{i+1} - f'_{i+1}"] \arrow[dl, "h_{i+1}"'] & P_i \arrow[r] \arrow[d, "f_i - f'_i"] \arrow[dl, "h_i"'] & \cdots \\
\cdots \arrow[r] & Q_{i+1} \arrow[r] & Q_i \arrow[r] & \cdots
\end{tikzcd}$$

satisfying $\partial h + h \partial = f - f'$.

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