Computations of Ext/Tor:

Let's compute, for abelian groups H, G ($\mathbb{Z}$-modules),

$\mathrm{Ext}^{(1)}(H,G)$, where

- H free (i.e., projective): can use resolution: $0 \to \underset{0}{P_1} \to \underset{H}{P_0} \to H \to 0$

  $\Rightarrow$ Ext complex is:

  $$\underset{\deg 1}{0 \leftarrow 0} \leftarrow \underset{\deg 0}{\mathrm{Hom}(P_0, G)} \leftarrow 0, \quad \mathrm{Hom}(P_0,G) = \mathrm{Hom}(H,G)$$

  $\Rightarrow \mathrm{Ext}(H,G) = 0.$

- $H = \mathbb{Z}/n\mathbb{Z}$, use

  $$0 \to (\underset{P_1}{\mathbb{Z}} \xrightarrow[f]{\times n} \underset{P_0}{\mathbb{Z}}) \to \mathbb{Z}/n\mathbb{Z} \to 0$$

  $\rightsquigarrow$ Ext chain complex: $0 \leftarrow \underset{G}{\mathrm{Hom}(\mathbb{Z}, G)} \xleftarrow[\delta^0 = f^*]{\times n} \underset{G}{\mathrm{Hom}(\mathbb{Z}, G)} \leftarrow 0$

  $\Rightarrow \mathrm{Ext}(H,G) = \mathrm{coker}(\delta^0) = G/nG.$

  $\begin{cases} \bullet\ \mathrm{Ext}(\mathbb{Z}/n, \underset{G}{\mathbb{Z}}) = \mathbb{Z}/n \\ \bullet\ \mathrm{Ext}(\mathbb{Z}/n, \mathbb{Q}) = 0 \end{cases}$

- $H = H_1 \oplus H_2$, then we can add projective resolutions of H_1 & H_2 to get one for H

  $\Rightarrow \mathrm{Ext}(H_1 \oplus H_2, G) = \mathrm{Ext}(H_1, G) \oplus \mathrm{Ext}(H_2, G).$

Cor (by classification of fin. gen. abelian groups):

For any fin. gen. abelian group H, $\mathrm{Ext}(H, \mathbb{Z}) = \mathrm{Tors}(H)$ torsion subgp.

(& $\mathrm{Hom}(H, \mathbb{Z}) = \mathrm{Free}(H)$ free subgrp.)

Rmk: Homology UCT involving Tor has a similar proof.

Note: • $\mathrm{Tor}^{(\mathbb{Z})}(\mathbb{Z}, G) := 0$ (use $(0 \to \underset{P_\cdot}{\mathbb{Z}}) \xrightarrow{\simeq} \mathbb{Z}$)

• $\mathrm{Tor}(\mathbb{Z}_m, G)$ (use $\underset{P_\cdot}{(\mathbb{Z} \xrightarrow{\times m} \mathbb{Z})} \xrightarrow{\simeq} \mathbb{Z}_m$

$$P_\bullet \otimes_{\mathbb{Z}} G := \underset{\deg 1}{G} \xrightarrow{\times m} \underset{\deg 0}{G}, \quad \text{so}$$

$$\mathrm{Tor}_{(0)} = \mathbb{Z}_m \otimes G = G/mG$$

$$\mathrm{Tor}_{(1)} = \ker(\times m) = \{m\text{-torsion subgroup of } G\}..$$

• exercise: $\mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}_n)$ & $\mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}_n)$.

For simplicity, we'll now focus on cohomology case of UCT:

<u>Theorem</u>: (UCT for cohomology) for any <u>free</u> chain complex $C_\bullet$, there is a (functorial) <u>natural</u> in $C_\bullet$ and G SES for each n:

$$0 \to \mathrm{Ext}^{(1)}_{(\mathbb{Z})}(H_{n-1}(C_\bullet), G) \longrightarrow H^n(\mathrm{Hom}(C_\bullet, G)) \xrightarrow{\beta} \mathrm{Hom}_{\mathbb{Z}}(H_n(C_\bullet), G) \to 0$$

Furthermore, this sequence splits (naturally in G, but <u>not</u> naturally in $C_\bullet$).

Recall β is the map $\quad [\gamma] \longmapsto ([\sigma] \longmapsto \gamma(\sigma))$.

In particular, this applies to compute $H^n(X;G) := H^n(\mathrm{Hom}_{\mathbb{Z}}(C_\bullet(X), G))$ in terms of $H_\bullet(X) := H_\bullet(C_\bullet(X))$.

e.g., using UCT there's a non-canonical isomorphism

$$H^n(X;\mathbb{Z}) \underset{\text{non-can.}}{\cong} \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ \mathrm{Hom}(H_n(X),\mathbb{Z})}}{\mathrm{Free}(H_n(X))} \oplus \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ \mathrm{Ext}(H_{n-1}(X),\mathbb{Z})}}{\mathrm{Tors}(H_{n-1}(X))}.$$

<u>Example</u>: $X = \mathbb{RP}^3$ recall that can compute $H_\bullet$ via <u>cellular chains</u>:

$$C_\bullet^{CW} = \{ \underset{\deg 0}{\mathbb{Z}} \xleftarrow{\times 0} \underset{\deg 1}{\mathbb{Z}} \xleftarrow{\times 2} \underset{\deg 2}{\mathbb{Z}} \xleftarrow{\times 0} \underset{\deg 3}{\mathbb{Z}} \}$$

$$\Rightarrow H_i(\mathbb{RP}^3;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}/2 & i=1 \end{cases} \qquad \Rightarrow H^i(\mathbb{RP}^3;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i=0,3 \\ \mathbb{Z}/2 & i=2 \end{cases}$$

$$\begin{cases} 0 & i=2 \\ \mathbb{Z} & i=3 \\ 0 & \text{else.} \end{cases} \qquad \text{UCT} \longrightarrow \qquad \Big| \; 0 \quad \text{else.}$$

(free part stays in same degrees, torsion part goes up in degree),

(note: by cohomological version of the $C_\bullet^{CW} \underset{\text{ch. htpy equiv}}{\simeq} C_\bullet^{sing}$ argument, we can compute that $C^\bullet_{sing} \equiv C^\bullet_{CW} \overset{?2}{\,} \mathrm{Hom}_{\mathbb{Z}}(C_\bullet^{CW}, \mathbb{Z})$),

(exercise: verify why this is true, and check in this case that $H^\bullet(\mathrm{Hom}(C_\bullet^{CW}, \mathbb{Z}))$ agrees with the answer above.

Proof of UCT (cohomology case, homology case requires similar argument):

Let $Z_n := \ker d_n$ cycles and $B_n := \mathrm{im}(d_{n+1})$ boundaries, so $B_n \subseteq Z_n \subseteq C_n$ and $Z_n/B_n = H_n$ homology.

we have short-exact sequences:

$$☆ \quad 0 \to B_n \xrightarrow{i_n} Z_n \xrightarrow{\pi_n} H_n \to 0$$

★★ (note: this is by hypothesis a free e.g. projective resolution of H_n. $\Rightarrow$ can compute $\mathrm{Ext}^i(H_n, M)$ as cohomologies $\Rightarrow \mathrm{Hom}(Z_n, M) \xrightarrow{i_n^*} \mathrm{Hom}(B_n, M) \to 0$ $\Rightarrow \mathrm{Ext}^1 = \mathrm{coker}(i_n^*)$.)

$$☆☆ \quad 0 \to Z_n \xrightarrow{j_n} C_n \underset{\sigma_n}{\overset{d_n}{\rightleftarrows}} B_{n-1} \to 0$$

can choose splitting b/c B_{n-1} free.

Applying $\mathrm{Hom}(-, M)$, we get:

(★) ($d_{n+1}: C_{n+1} \to C_n$ factors through $C_{n+1} \to B_n \overset{i_n}{\hookrightarrow} Z_n \hookrightarrow C_n$).

$$\begin{tikzcd}
 & & 0 & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}(H_n, M) \arrow[r, "\pi_n^*"] & \mathrm{Hom}(Z_n, M) \arrow[u] \arrow[r, "i_n^*"] \arrow[dr, phantom, "★_1"] & \mathrm{Hom}(B_n, M) \arrow[d, "d_{n+1}^*"] \\
 & \mathrm{Hom}(C_{n-1}, M) \arrow[r, "d_n^*"] \arrow[d, "j_{n-1}^*"] \arrow[dr, phantom, "★_2"] & \mathrm{Hom}(C_n, M) \arrow[u, "j_n^*"] \arrow[r, "d_{n+1}^*"] \arrow[d, dashed, bend right, "\sigma_n^*"'] & \mathrm{Hom}(C_{n+1}, M) \\
 & \mathrm{Hom}(Z_{n-1}, M) \arrow[r, "i_{n-1}^*"] \arrow[d] & \mathrm{Hom}(B_{n-1}, M) \arrow[u, "d_n^*"] \arrow[r, two heads, "\delta"] & \mathrm{Hom}(B_{n-1}) / i_{n-1}^*(\mathrm{Hom}(Z_{n-1}, M)) \arrow[u, dashed, "?\ 0"] \\
 & 0 & 0 \arrow[u] &
\end{tikzcd}$$

$\mathrm{Hom}(B_{n-1}) / i_{n-1}^*(\mathrm{Hom}(Z_{n-1}, M)) = \mathrm{Ext}^1_R(H_{n-1}, M)$ by ★★

Observations:

(1) The map $\beta : \dfrac{\ker d_{n+1}^*}{\operatorname{im} d_n^*} \longrightarrow \operatorname{Hom}(H_n, M) \left(= \operatorname{Hom}(Z_n, M)_{\operatorname{Ann}(B_n)}\right)$

$$[\gamma] \longmapsto \left([z] \mapsto \gamma(z)\right)$$

well-defined β only depends on $[\gamma], [z]$.)

look at map induced by any rep. γ on cycles & check it annihilates boundary, they also result only depends on $[\gamma]$

can be understood as follows: $H_n = Z_n / B_n$ and $Z_n \xrightarrow{\pi_n} H_n$ induces the map $\pi_n^* : \operatorname{Hom}(H_n, M) \to \operatorname{Hom}(Z_n, M)$ whose image $\operatorname{im}(\pi_n^*)$ consists of those $Z_n \to M$ which are zero along $B_n \subseteq Z_n$, i.e., annihilate B_n.

Now note $\ker(d_{n+1}^* : \operatorname{Hom}(C_n, M) \to \operatorname{Hom}(C_{n+1}, M)) = \ker(i_n^* j_n^*)$

(by comm. diagram $\star_1$ & injectivity of $\operatorname{Hom}(B_n, M) \xrightarrow{d_{n+1}^*} \operatorname{Hom}(C_{n+1}, M)$.)

Similarly, since j_{n-1}^* surjective, $\star_2$ implies $\operatorname{im}(d_n^*) = \operatorname{im}(d_n^* \circ i_{n-1}^*)$, so

$$\frac{\ker(d_{n+1}^*)}{\operatorname{im}(d_n^*)} = \frac{\ker(i_n^* j_n^*)}{\operatorname{im}(d_n^* i_{n-1}^*)}$$

Now, the map

$$\frac{\ker(i_n^* j_n^*)}{\operatorname{im}(d_n^* i_{n-1}^*)} \xrightarrow{[j_n^*]} \frac{\ker(i_n^*)}{\operatorname{im}(j_n^* d_n^* i_{n-1}^*) = 0} = \operatorname{im}(\pi_n^*) \xrightarrow[\cong]{(\pi_n^*)^{-1}} \operatorname{Hom}(H_n, M)$$

(b/c $j_n^* d_n^* = 0$)

is precisely β. (that is, takes a class $[\gamma]$, for any rep. $\gamma \in \ker i_n^* j_n^*$, apply j_n^* to it to get an element of $\operatorname{Hom}(Z_n, M)$ which annihilates B_n hence lies in $\operatorname{im}(\pi_n^*)$.

Now, j_n^* was surjective, hence β is also.

kernel of β? (by vertical SES, $\operatorname{im}(d_n^*) = \ker j_n^* \subseteq \ker(i_n^* j_n^*)$)

hence $\ker(\beta) = \ker[j_n^*] = \dfrac{\operatorname{im}(d_n^*)}{\operatorname{im}(d_n^* i_{n-1}^*)}$

$$\operatorname{im}(d_n^* i_{n-1}^*)$$

$$\underset{(\text{as } d_n^* \text{ injective})}{\overset{(d_n^*)^{-1}}{\cong}} \frac{\operatorname{Hom}(B_{n-1}, M)}{\operatorname{im}(i_{n-1}^*)} \xrightarrow[\delta]{\cong} \operatorname{Ext}_R^1(H_{n-1}, M).$$

Hence get the desired SES:

$$0 \to \ker(\beta) \to H^n(\{\operatorname{Hom}(C_i, M), d_i^*\}) \to \operatorname{Hom}(H_n, M) \to 0$$

$$\ker(\beta) \cong \operatorname{Ext}_R^1(H_{n-1}, M).$$

Splitting?

Use $\delta \sigma_n^* : \operatorname{Hom}(C^n, M) \to \operatorname{Ext}_R^1(H_{n-1}, M)$, which

induces $H^n \xrightarrow{s} \operatorname{Ext}_R^1(H_{n-1}, M)$, splitting

$$0 \to \operatorname{Ext}_R^1(H_{n-1}, M) \longrightarrow H^n \to \operatorname{Hom}(H_n, M) \to 0$$

s.

—————— lecture end ——————

UCT over more general rings:

Thm (UCT/R): R any PID (e.g., $\mathbb{Z}$, any field), and $C_\bullet$ a chain complex of free R-modules, G another R-module. Then, $\exists$ SES

$$0 \to \operatorname{Ext}_{(R)}^{(1)}(H_{n-1}(C_\bullet), G) \to H^n(\operatorname{Hom}_R(C_\bullet, G)) \xrightarrow{\beta} \operatorname{Hom}_R(H_n(C_\bullet), G) \to 0$$

natural in $C_\bullet$ and G, & split (not naturally split).

(R PID $\Rightarrow$ $0 \to B_n \to Z_n \to H_n \to 0$ gives a proj. resolution of H_n, for instance).

In particular, if we begin with $C_\bullet(X;R)$ ($:= C_\bullet(X) \otimes_{\mathbb{Z}} R$), and

$$C^\bullet(X;R) := \operatorname{Hom}_{\mathbb{Z}}(C_\bullet(X), R) \cong \operatorname{Hom}_R(C_\bullet(X;R), R) \quad (\text{why?})$$

In particular, we can now compute $H^\bullet(X;R)$ in terms of $H_\bullet(X;R)$ using UCT/R.

Special case: $R = k$ a field (i.e., $\mathbb{Q}$, $\mathbb{Z}/2\mathbb{Z}$, etc.) then any k-module M is automatically free hence projective. $\Rightarrow \mathrm{Ext}^{(1)}_k(M,k) = 0$ (b/c $(0 \to M) \xrightarrow{\simeq} M$ is a proj. resolution)

$$\Rightarrow \boxed{H^n(X;k) \underset{\cong}{\xrightarrow{\beta}} \mathrm{Hom}_k(H_n(X;k),k) = H_n(X;k)^\vee} \quad \boxed{\text{(over a field)}}.$$