Last time's proof directly generalizes to (same exact proof)

<u>Thm</u> ($UCT/_R$): R any PID (e.g., $\mathbb{Z}$, any field), and $C_\bullet$ a chain complex of free R-modules, G another R-module. Then, $\exists$ SES

$$0 \to \mathrm{Ext}^{(1)}_{(R)}(H_{n-1}(C_\bullet), G) \to H^n(\mathrm{Hom}_R(C_\bullet, G)) \xrightarrow{\beta} \mathrm{Hom}_R(H_n(C_\bullet), G) \to 0$$

natural in $C_\bullet$ and G, & split (not naturally split).

(R PID $\Rightarrow$ $0 \to B_n \to Z_n \to H_n \to 0$ gives a proj. resolution of H_n, for instance).

In particular, if we begin with $C_\bullet(X;R)$ ($:= C_\bullet(X) \otimes_{\mathbb{Z}} R$), and

$$C^\bullet(X;R) := \mathrm{Hom}_{\mathbb{Z}}(C_\bullet(X), R) \cong \mathrm{Hom}_R(C_\bullet(X;R), R) \quad (\text{why?})$$

In particular, we can now compute $H^\bullet(X;R)$ in terms of $H_\bullet(X;R)$ using UCT/R.

<u>Special case</u>: $R = k$ a field (i.e., $\mathbb{Q}$, $\mathbb{Z}/2\mathbb{Z}$, etc.) then any k-module M is automatically free hence projective. $\Rightarrow \mathrm{Ext}^{(1)}_k(M,k) = 0$ (b/c $(0 \to M) \xrightarrow{\simeq} M$ is a proj. resolution)

$$\Rightarrow \boxed{H^n(X;k) \underset{\cong}{\xrightarrow{\beta}} \mathrm{Hom}_k(H_n(X;k), k) = H_n(X;k)^\vee} \quad \boxed{\text{(love a field)}}.$$

<u>Künneth theorems in homology and cohomology</u>

<u>Goal</u>: understand relationship between $H_\bullet / H^\bullet$ of $X \times Y$ and $H_\bullet / H^\bullet$ of individual factors.

<u>over a field</u>, the result will state

tensor of graded abelian groups

- $H_\bullet(X \times Y; k) \cong H_\bullet(X;k) \otimes H_\bullet(Y;k)$

  (means $H_n(X \times Y; k) \cong \bigoplus_{i+j=n} H_i(X;k) \otimes H_j(Y,k)$.)

- similar for cohomology, assuming at least one of X, Y <u>finite type</u>

  (Z finite type if each $H_i(Z;k)$ is finitely generated). (e.g., CW cplx w/ finitely many cells in each dimension)

  (basic problem is that $(V \otimes W)^\vee \not\cong V^\vee \otimes W^\vee$ in general, it is if one of V, W is fin. dim'l).

- In general over R, there's a map which fails to be an $\cong$ (coker is $\mathrm{Tor}(-,-)$)

Künneth is an immediate consequence of two results:

today → (1) The Eilenberg–Zilber theorem says $C_\bullet(X\times Y) \underset{\text{ch. htpy equiv}}{\xrightarrow{\simeq}} \underbrace{C_\bullet(X)\otimes C_\bullet(Y)}$ can take ⊗ of chain cplxes & get a chain complex.

(2) The algebraic Künneth theorem comparing
$H_\bullet(C_\bullet \otimes D_\bullet)$ to $H_\bullet(C_\bullet)\otimes H_\bullet(D_\bullet)$ (a Tor term appears).
generalizes Homology UCT in a way by allowing $D_\bullet$ to not just be R.

(reference: [Bredon])

Def: $C_\bullet$ and $D_\bullet$ chain complexes over R ($=\mathbb{Z}$ for now);
define $C_\bullet \otimes_{(R)} D_\bullet$ by $(C_\bullet\otimes D_\bullet)_n = \bigoplus_{i+j=n} C_i \otimes D_j$, with   ← tensor of graded abelian groups

$$\partial_{C_\bullet\otimes D_\bullet}(a\otimes b) = \partial a\otimes b + (-1)^{\deg(a)=i}\, a\otimes \partial b.$$

degree i degree j

can think of this as $\partial_{C_\bullet\otimes D_\bullet} = \partial\otimes \mathrm{id} + \mathrm{id}\otimes\partial$, using the convention
that $(f\otimes g)(a\otimes b) = (-1)^{\deg(g)\deg(a)} f(a)\otimes g(b)$

Recall, a chain homotopy equivalence between $A_\bullet$ and $B_\bullet$ consists of

$$A_\bullet \underset{g}{\overset{f}{\rightleftarrows}} B_\bullet \qquad f, g \text{ chain maps } \left(\text{e.g., } \begin{array}{l} f\circ\partial_A = \partial_B\circ f \\ g\circ \partial_B = \partial_A\circ g\end{array}\right)$$

with $f\circ g \underset{\text{ch. htpic}}{\simeq} \mathrm{id}_{B_\bullet}$ $g\circ f \underset{\text{ch. htpic}}{\simeq} \mathrm{id}_{C_\bullet}$

$\Rightarrow$ $[f]$, $[g]$ induce inverse isos. on $H_\bullet(A) \rightleftarrows H_\bullet(B)$.

Theorem: (Eilenberg–Zilber): There is a chain homotopy equivalence (over any coeffs. R)

θ ← (particular choice often called) Alexander–Whitney map

$$C_\bullet(X\times Y) \underset{(-)\times(-)}{\overset{\theta}{\underset{\simeq}{\rightleftarrows}}} C_\bullet(X)\otimes C_\bullet(Y),$$

which is natural (functorial in X and Y), are unique up to chain homotopy.

"cross product"
(specific model often called) (Eilenberg–Zilber map).

To start, we need to define the maps. Let's begin with the cross product

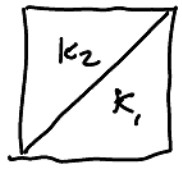
$$\times : C_p(X)\otimes C_q(Y) \longrightarrow C_{p+q}(X\times Y).$$

How to define?

Given a generator $\sigma : \Delta^p \to X$, $\tau : \Delta^q \to Y$, want "$\sigma \times \tau$" $\in C_{p+q}(X \times Y)$.

- Take the naive product $(\sigma, \tau) : \Delta^p \times \Delta^q \longrightarrow X \times Y$.
- if $p = 0$ or $q = 0$ then $\Delta^p \times \Delta^q \cong \Delta^{p+q}$ $\quad (\Delta^p \times \Delta^0 = \Delta^p)$.

  in this case, define $\sigma \times \tau := (\sigma, \tau)$



$\Delta^1 \times \Delta^1$ can be triangulate.

- In general, $\Delta^p \times \Delta^q$ is not a simplex, but it can be triangulated $\Delta^p \times \Delta^q = \bigcup \kappa_i : \Delta^{p+q} \to \Delta^p \times \Delta^q$

  & $\overset{\text{roughly}}{\text{define}}$ $\sigma \times \tau := \sum (\sigma, \tau)|_{\kappa_i}$.

<u>Special case</u>: 'prism operator' involves triangulating $\Delta^p \times \Delta^1$ for all p.

(used to show $f \simeq g \Rightarrow f_\# \underset{\text{ch. htpc}}{\simeq} g_\#$).

<u>options</u>: has some advantages too (e.g., $\times$ is strictly associative)

- explicit formula (combinatorial, generalizes 'prism', get one $p+q$ simplex for each "shuffle" of $(v_0, \dots, v_p)$ & $(w_0, \dots, w_q)$ vertices of Δ^p & Δ^q)

we'll take this approach

- argue that such a map has to exist for general reasons, using "<u>method of acyclic models</u>" (proof technique used a lot in comparing homology theories: singular vs. simplicial vs. cubical etc.)

<u>Thm</u> (existence of $\times$): For each p, q, $\exists$ bilinear

$$\times : C_p(X) \times C_q(Y) \longrightarrow C_{p+q}(X \times Y) \quad \text{such that:}$$

(1) For $x_0 : \Delta^0 \to X$, $x_0 \times \tau = (x_0, \tau) : \Delta^{0+q} = \Delta^q \longrightarrow X \times Y$

Similarly, for $y_0 : \Delta^0 \to Y$, $\sigma \times y_0 = (\sigma, y_0)$.

(2) (naturality): If $f : X \to X'$ $g : Y \to Y'$ induces $(f,g) : X \times Y \to X' \times Y'$, then $(f,g)_\# (a \times b) = (f_\# a) \times (f_\# b)$.

(3) (<u>chain map / boundary formula</u>): $\times$ is a chain map $C_\cdot(X) \otimes C_\cdot(Y) \longrightarrow C_\cdot(X \times Y)$.

$$\partial(a \times b) = \partial a \times b + (-1)^{\deg(a)} a \times \partial b.$$

<u>Pf</u>: Induction on p, q.

- base case: have such maps when $p=0$ or $q=0$.

- Inductive step: fix $p>0$ and $q>0$ (so $p+q \geq 1$) & say $\times$ has been defined for all smaller $(p+q)$'s for all X and Y.

Want to define $\sigma \times \tau$ for $\sigma \in C_p(X)$, $\tau \in C_q(Y)$.

First define $\times$ on a very special p-simplex $\times$ a very special singular q-simplex in special spaces: namely consider

$$i_p : \Delta^p \xrightarrow{id} \Delta^p \qquad \rightsquigarrow \text{ give elements in } C_p(\Delta^p) \text{ \& } C_q(\Delta^q)$$
$$i_q : \Delta^q \xrightarrow{id} \Delta^q \qquad \text{respectively.}$$

Let's try to first define $i_p \times i_q \in C_{p+q}(\Delta^p \times \Delta^q)$. How?

By (3) we want $i_p \times i_q$ to satisfy:

call this expression α.

$$\underline{(*)} \quad \partial(\overbrace{\underbrace{i_p \times i_q}_{\text{not yet defined}}}^{\beta}) = \overbrace{\partial i_p \times i_q + (-1)^p i_p \times \partial i_q}^{\alpha}.$$

both inductively defined, as we've defined $\times$ on all $C_k(X) \otimes C_\ell(Y)$ for $k+\ell < p+q$.

Compute $\partial(\text{RHS}) = \partial(\alpha)$:

$$= \cancel{\partial\partial i_p} \times i_q + (-1)^{p-1} \partial i_p \times \partial i_p + (-1)^p \partial i_p \times \partial i_q + i_p \times \cancel{\partial\partial i_q} = 0$$

cancel.

So in fact α is a cycle in $C_{p+q-1}(\Delta^p \times \Delta^q)$.

We want $\alpha = \partial\beta$, i.e., want α to be a boundary.

"acyclic"

Since $p+q-1 > 0$ and $\Delta^p \times \Delta^q$ is contractible, $H_{p+q-1}(\underline{\Delta^p \times \Delta^q}) = 0$, so in fact $\exists$ a chain β with $\partial\beta = \alpha$.

Pick any such chain & call it $i_p \times i_q$.

What to do for a general $\sigma : \Delta^p \to X$, $\tau : \Delta^q \to Y$? In fact, $\sigma \times \tau$ is forced by naturality: note that as an element of $\sigma \in C_p(X)$, $\sigma = \sigma_\# \circ i_p$, $\sigma_\# : C_p(\Delta^p) \to C_p(X)$, $i_p \in C_p(\Delta^p)$.

$$\begin{tikzcd} \Delta^p \arrow[r, "i_p = id"] \arrow[rr, bend right, "\sigma"'] & \Delta^p \arrow[r, "\sigma"] & X \end{tikzcd}$$

Similarly, $\tau = \tau_\# \circ i_q$.

Hence if $(\sigma,\tau):\Delta^p\times\Delta^q\to X\times Y$ is the product map, by naturality (2), we get:

$$\underset{\text{(if defined)}}{\sigma\times\tau} = (\sigma_\# i_p)\times(\tau_\# i_q) \underset{\text{naturality}}{=\!=} (\sigma,\tau)_\#\underbrace{(i_p\times i_q)}$$

defined above!

In order for naturality to hold, we must define $\sigma\times\tau := (\sigma,\tau)_\#(i_p\times i_q)$.

($i_p:\underline{\Delta^p\to\Delta^p}$, $i_q:\underline{\Delta^q\to}\ \underline{\Delta^q}$ are the "models").

Check that this definition satisfies the boundary formula:

compute $\partial(\sigma\times\tau) = \partial((\sigma,\tau)_\#(i_p\times i_q)) = (\sigma,\tau)_\#(\partial(i_p\times i_q))$

$$\underset{\substack{\text{boundary}\\ \text{formula for } i_p\times i_q}}{=\!=} (\ \cdots\ ) \underset{\text{(exercise).}}{=} \cdots = \partial\sigma\times\tau+(-1)^{\deg(\sigma)}\sigma\times\partial\tau.$$

$\boxtimes$