Last time: Thm: $\exists$ map $\times : C_\bullet(X) \otimes C_\bullet(Y) \to C_\bullet(X \times Y)$ s.t.

(1) $x_0 \times \tau = (x_0, \tau)$ $\quad \tau \times y_0 = (\tau, y_0)$ for $x_0 \in C_0(X), y_0 \in C_0(Y)$

(2) natural in X, Y

(3) chain map.

Note: For pairs, define $(X,A) \times (Y,B) = (X \times Y, X \times B \cup A \times Y)$.

check/note that $\times$ naturally takes $C_\bullet(X,A) \otimes C_\bullet(Y,B)$ into $C_\bullet((X,A) \times (Y,B))$.

(e.g., $A \subset X$, then $\times : C_\bullet(X) \otimes C_\bullet(Y) \to C_\bullet(X \times Y)$ carries $C_\bullet(A) \otimes C_\bullet(Y)$ to $C_\bullet(A \times Y)$).

The map θ (the other way):

Technical lemma: Say X, Y are contractible, then $C_\bullet(X) \otimes C_\bullet(Y)$ is acyclic

i.e., $H_n(-) \begin{cases} = 0 & \text{for } n > 0 \\ = \mathbb{Z} & \text{for } n = 0, \text{ generated by } [x_0 \otimes y_0], \end{cases}$

$x_0 : \Delta^0 \to X$, $y_0 : \Delta^0 \to Y$ any points.

Pf sketch: X contractible $\iff$ $\{*\} \underset{pr}{\overset{x_0}{\rightleftarrows}} X$ are homotopy inverse, in particular

$$\begin{tikzcd} X \arrow[r, "pr"] \arrow[rr, bend right, "\varepsilon_X"'] & \{*\} \arrow[r, "x_0"] & X \end{tikzcd}$$

ε_X is homotopic to id_X.

similarly $\varepsilon_Y : Y \xrightarrow{pr} \{*\} \xrightarrow{y_0} Y$ is htpic to id_Y.

$\Rightarrow \exists$ chain homotopies H_X (on $C_\bullet(X)$) between $(\varepsilon_X)_\#$ and $id_{C_\bullet(X)} = (id_X)_\#$ (deg +1)

H_Y (on $C_\bullet(Y)$) between $(\varepsilon_Y)_\#$ and $id_{C_\bullet(Y)} = (id_Y)_\#$. (deg +1)

i.e., $\partial H_X + H_X \partial = id - (\varepsilon_X)_\#$, same for H_Y, ε_Y.

☆ ($\overset{e.g.,}{\Rightarrow}$ $[id] = (\varepsilon_X)_* : H_\bullet(X) \to H_\bullet(*) \to H_\bullet(X)$ $\Rightarrow$ $H_\bullet(X) = 0$ in deg > 0

X connected so $H_0(X) = \mathbb{Z}$)

same for Y,

Let $H_\otimes := H_X \otimes id_{C_\bullet(Y)} + (\varepsilon_X)_\# \otimes H_Y$ on $C_\bullet(X) \otimes C_\bullet(Y)$ (deg +1 map)

"contract X to a point" $\quad$ already contracted id_X to ε_X, now contract Y down.

Exercise: $\partial_{C_\bullet(X) \otimes C_\bullet(Y)} H_\otimes + H_\otimes \circ \partial_{C_\bullet(X) \otimes C_\bullet(Y)} = id \otimes id - \varepsilon_X \otimes \varepsilon_Y$.

• finish the proof from here, using analogue of ☆ . ☒.

Thm: (existence of Θ): $\exists$ $\Theta: C_\bullet(X \times Y) \to C_\bullet(X) \otimes C_\bullet(Y)$ satisfying:

(1) Θ is a chain map.

(2) Θ is natural in X & Y

(i.e, $f: X \to X'$, $g: Y \to Y'$ then $(f,g): X \times Y \to X' \times Y'$ and $\underline{\Theta \circ (f,g)_\# = (f_\# \otimes g_\#) \circ \Theta}$.).

(3) In degree 0, Θ is the following (determined) map:

$$\underbrace{\{(x,y): \Delta^0 \cong \Delta^0 \times \Delta^0 \to X \times Y\}}_{\cap \; C_0(X \times Y)} \longmapsto \underbrace{(x: \Delta^0 \to X) \otimes (y: \Delta^0 \to Y)}_{\cap \; C_0(X) \otimes C_0(Y)}.$$

Pf: again induction, appealing to the method of acyclic models.

• base case: (deg 0): defined by (3).

• say Θ defined in degrees $< k$, is a (natural) chain map, for all X, Y. To define Θ in degree k, first consider the special case

$X = \Delta^k = Y$, with special singular simplex $d_k: \Delta^k \xrightarrow{(id, id)} \overset{X}{\Delta^k} \times \overset{Y}{\Delta^k}$ diagonal singular simplex.

so $d_k \in C_k(\Delta^k \times \Delta^k)$.

By induction, we've defined

$$\Theta(\partial d_k) \in \underbrace{(C_\bullet(\Delta^k) \otimes C_\bullet(\Delta^k))_{k-1}}, \text{ and we can check directly that}$$

claim: $\Theta(\partial d_k)$ is a cycle in $\uparrow$. Follows from $\partial_\otimes \Theta(\partial d_k) \underset{\substack{\Theta \text{ chain map} \\ \text{in degree } k-1}}{=} \Theta(\overset{0}{\cancel{\partial \partial}} d_k) = 0.$

We are seeking to define a $\Theta(d_k)$ chain satisfying chain eq'n,

$$\partial(\underbrace{\Theta(d_k)}_{\substack{\nearrow \\ \text{not yet defined;}}}) = \underbrace{\Theta(\partial d_k)}_{\text{defined inductively, } \& \text{ is a cycle by above}}$$

but if $[\Theta(\partial d_k)] = 0$ in $H_{k-1}(C_\bullet(\Delta^k) \otimes C_\bullet(\Delta^k))$, can pick any chain β w/ $\boxed{\partial\beta = \Theta(\partial d_k)}$ β set $\Theta(d_k) = \beta$. (choice).

$\star$

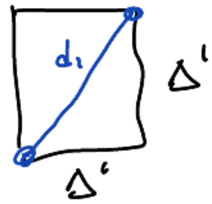
If $k > 1$, technical lemma $\Rightarrow H_{k-1}(C_\bullet(\Delta^k) \otimes C_\bullet(\Delta^k)) = 0$ b/c Δ^k, Δ^k acyclic.

$\Rightarrow$ such a β exists.

If $k = 1$, $H_0(C_\bullet(\Delta^k) \otimes C_\bullet(\Delta^k)) = \mathbb{Z}$, but we can directly compute that

$[\Theta(\partial d_1)] = 0$, therefore a β exists.

$\Delta^1 \times \Delta^1$



$$\left(\begin{array}{l} \parallel \\ [\Theta((x_1,y_1) - (x_0,y_0))] \\ \parallel (3) \\ [x_1 \otimes y_1 - x_0 \otimes y_0] \\ = 0 \text{ by technical lemma.} \end{array} \right)$$

$\Theta(d_k) :=$ any choice of such β satisfying $\star$.

General X, Y, $\sigma : \Delta^k \longrightarrow X \times Y$ singular simplex:

notation: $\pi_X : X \times Y \to X$ projection, resp. $\pi_Y : X \times Y \longrightarrow Y$.

Note that $\pi_X\sigma : \Delta^k \to X$, $\pi_Y\sigma : \Delta^k \to Y$ gives

$(\pi_X\sigma, \pi_Y\sigma) : \Delta^k \times \Delta^k \longrightarrow X \times Y$, w/ σ factoring as

$$\Delta^k \xrightarrow[(id,\, id)]{d_k} \Delta^k \times \Delta^k \xrightarrow{(\pi_X\sigma, \pi_Y\sigma)} X \times Y.$$

So $\sigma = (\pi_X\sigma, \pi_Y\sigma)_\# d_k$.

Hence by naturality, $\Theta(\sigma)$ should satisfy:

$$\boxed{\Theta(\sigma) = \Theta((\pi_X\sigma, \pi_Y\sigma)_\# d_k) = \underbrace{((\pi_X\sigma) \otimes (\pi_Y\sigma))}_{\text{defined.}}\underbrace{(\Theta(d_k))}_{\text{already defined.}}}$$

Hence, we can simply use $\uparrow$ to define $\Theta(\sigma)$.

Exercise: check this def'n satisfies (1)→(3) in particular (1) & (2). $\boxtimes$.

we've defined $\Theta : C_\bullet(X \times Y) \to C_\bullet(X) \otimes C_\bullet(Y)$ and $\times : C_\bullet(X) \otimes C_\bullet(Y) \to C_\bullet(X \times Y)$

Q: are they homotopy inverses? what if we made different 'choices' of Θ, $\times$ (by choosing different bounding chains?)

Thm: Any two natural chain maps, either $\boxed{\text{in deg } 0}$

- $id_0 : C_0(X \times Y) \to C_0(X \times Y)$ • from $C_\bullet(X \times Y)$ to itself (e.g., $id_{C_\bullet(X \times Y)}$, $(- \times -) \circ \Theta$)
- $id_0 : C_0(X) \otimes C_0(Y) \to C_0(X) \otimes C_0(Y)$ • from $C_\bullet(X) \otimes C_\bullet(Y)$ to itself (e.g., $\Theta \circ (- \times -)$, $id_{C_\bullet(X) \otimes C_\bullet(Y)}$)
- $(x,y) : \Delta^0 \to X \times Y \mapsto \{x\} \otimes \{y\}$ • from $C_\bullet(X \times Y)$ to $C_\bullet(X) \otimes C_\bullet(Y)$, (e.g., Θ, another choice Θ')
- $\{x\} \otimes \{y\} \longmapsto (x,y)$. • from $C_\bullet(X) \otimes C_\bullet(Y)$ to $C_\bullet(X \times Y)$ ($\times$, another choice of $\times'$).

fixed

that coincide w/ the canonical maps in degree 0, are chain homotopic.

Cor: Eilenberg-Zilber theorem as stated: $\exists$ natural chain htpy equiv. $C_\cdot(X\times Y) \underset{\times}{\overset{\Theta}{\rightleftarrows}} C_\cdot(X)\otimes C_\cdot(Y)$,
w/ Θ, $\times$ unique up to chain homotopy.

Pf sketch of theorem: All 4 cases are similar, & all use method of acyclic models
use the "models" • $i_p\otimes i_q \in C_p(\Delta^p)\otimes C_q(\Delta^q)$ when starting from $C_\cdot(X)\otimes C_\cdot(Y)$
• $d_p \in C_p(\Delta^p\times\Delta^p)$ when starting from $C_\cdot(X\times Y)$.

In each case, given a pair ϕ, ψ of natural maps, coinciding in degree 0, try to construct a chain htpy D inductively satisfying $\partial D + D\partial = \phi - \psi$. Again inductively in each degree first construct D(model chain), then push forward. D(model chain) should satisfy

$$\partial D(\text{model chain}) = \phi(\text{model chain}) - \psi(\text{model chain}) - D\underbrace{\partial(\text{model chain})}_{\text{inductively already constructed.}}.$$

As long as we know RHS is a cycle, & relevant $H_\cdot(\text{model space})$ is either 0 or at least $[RHS]=0$ in $H_\cdot$, then a chain β satisfying $\partial\beta = RHS$ exists, & pick such a β & call it D(model chain).
Now 'push forward' to define D(any chain) b/c every chain is pushed forward from model.

Exercise: use this to spell out the details in 1-2 cases above. ∎.

Eilenberg-Zilber implies

recall coeffs. allowed in arguments above

$$H_p(X\times Y;R) \cong H_p(C_\cdot(X\times Y;R)) \underset{(EZ)}{\cong} \underline{H_p(C_\cdot(X;R)\otimes_R C_\cdot(Y;R))}$$

Q1: can we analyze RHS in terms of $\otimes$ of Homology groups?

|| (if one desires)

$$H_p(C_\cdot(X)\otimes_{\mathbb{Z}} C_\cdot(Y)\otimes_{\mathbb{Z}} R).$$

By dualizing on chain level, one gets ch. htpy equivalences:

R-coeffs.

$$\mathrm{Hom}_R(C_\cdot(X\times Y;R),R) \rightleftarrows \mathrm{Hom}_R(C_\cdot(X)\otimes C_\cdot(Y),R)$$
$$\|$$
$$C^\cdot(X\times Y;R)$$

$$\rightsquigarrow H^\cdot(X\times Y;R) \cong \underline{H^\cdot(\mathrm{Hom}_R(C_\cdot(X)\otimes C_\cdot(Y),R))}$$

Q2: how does this compare to $\otimes$ of cohomologies?

Regarding Q1, we have

generalizes homology UCT

e.g., $\mathbb{Z}$ or a field.

Thm: (Algebraic Künneth theorem): let $K_\cdot, L_\cdot$ free chain cplxes (over any PID R), then
$\exists$ a natural in $K_\cdot, L_\cdot$ SES; for each n

$$0 \longrightarrow \big(H_\bullet(K_\bullet)\otimes H_\bullet(L_\bullet)\big)_n \xrightarrow{\ \alpha\ } H_n(K_\bullet\otimes L_\bullet) \longrightarrow \mathrm{Tor}_{(1)}^{(R)}\big(H_\bullet(K_\bullet),H_\bullet(L_\bullet)\big)_{n-1} \to 0,$$

means $\bigoplus_{i+j=n} H_i(K_\bullet)\otimes H_j(L_\bullet)$

the standard map $[a]\otimes[b] \longmapsto [a\otimes b]$.

means $\bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(K_\bullet),H_j(L_\bullet))$.

& it splits (non-naturally).

Pf has the same idea as proof of cohomology UCT; study failure of α to be injective via analyzing elements of $\mathrm{coker}(\alpha)$ (instead of "$\ker(\beta)$"). (omitted).

Cor (of E-Z + Alg Künneth): Künneth theorem for homology: R PID, implicitly take R-coefficients. Then there is a natural SES (which splits, but non-naturally):

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(X)\otimes H_q(Y) \xrightarrow{[\times]} H_n(X\times Y) \longrightarrow \bigoplus_{i+j=n-1}\mathrm{Tor}_1^R(H_i(X),H_j(Y)) \to 0$$

$$H_n(X\times Y) \cong H_n(C_\bullet(X\times Y)) \xleftarrow[\cong]{\times\ \text{EZ}} H_n(C_\bullet(X)\otimes C_\bullet(Y))$$

map appearing in alg künneth.

If $R=k$ is a field, we know all Tor_1^k's are 0.

$\Rightarrow$ künneth isomorphism: $\boxed{[\times] : H_\bullet(X,k)\otimes H_\bullet(Y,k) \xrightarrow{\ \cong\ } H_\bullet(X\times Y;k).}$

Example: Compute $H_\bullet(\mathbb{RP}^3\times\mathbb{RP}^3,k)$, for k any field.

künneth: $\xleftarrow{\cong} H_\bullet(\mathbb{RP}^3,k)\otimes H_\bullet(\mathbb{RP}^3,k)$.

we know $\mathbb{RP}^3$ has CW homology chain complex (w/ R-coeffs):

$$\begin{array}{cccc} \deg 0 & \deg 1 & \deg 2 & \deg 3 \\ R \xleftarrow{\times 0} & R \xleftarrow{\times 2} & R \xleftarrow{\times 0} & R \end{array}$$

$$\Rightarrow H_i(\mathbb{RP}^3,R) = \begin{cases} R & i=0,3 \\ R/2R & i=1 \\ 2\text{-torsion}(R) & i=2 \\ 0 & \text{else} \end{cases} \overset{\text{over a field}}{=} k \quad \begin{cases} \mathrm{char}(k)=2 & \begin{cases} k & i=0,1,2,3 \\ 0 & \text{else.}\end{cases} \\ \mathrm{char}(k)\neq 2 & \begin{cases} k & i=0,3 \end{cases} \end{cases}$$

over $\mathbb{Z}$

$$\begin{cases} \mathbb{Z} & i=0,3 \\ \mathbb{Z}/2 & i=1 \\ 0 & i=2. \end{cases}$$

$\{ 0$ else.

in this case $\cong H_\bullet(S^3, k)$ ← char $k \neq 2$.

get, for char$(k)=2$:

$H_p(\mathbb{RP}^3) \otimes H_q(\mathbb{RP}^3)$ is:

p \ q	0	1	2	3
0	k	k	k	k
1	k	k	k	k
2	k	k	k	k
3	k	k	k	k

deg 0, deg 1, deg 2, deg 3, deg 4, deg 5, deg 6

$H_\bullet(\mathbb{RP}^3 \times \mathbb{RP}^3)$ is:

deg	
0	k
1	$k \oplus k$
2	$k \oplus k \oplus k$
3	$k \oplus k \oplus k \oplus k$
4	$k \oplus k \oplus k$
5	$k \oplus k$
6	k

char$(k) \neq 2$:

	0	1	2	3
0	k	0	0	k
1	0	0	0	0
2	0	0	0	0
3	k	0	0	k

$$\Rightarrow H_\bullet(\mathbb{RP}^3 \times \mathbb{RP}^3, k) = \begin{cases} k & i=0,6 \\ k \oplus k & i=3 \\ 0 & \text{else.} \end{cases}$$

Exercise: compute $H_i(\mathbb{RP}^3 \times \mathbb{RP}^3, \mathbb{Z})$. (using splitting of SES) There's a Tor term appearing, & in sum one gets:

$H_i \otimes_{\mathbb{Z}} H_j$

	0	1	2	3
0	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}$
1	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
2	0	0	0	0
3	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}$

only one Tor, $\mathrm{Tor}_1^{\mathbb{Z}}(\overset{H_1}{\mathbb{Z}/2}, \overset{H_1}{\mathbb{Z}/2}) \cong \mathbb{Z}/2$, contributes to $H_3(\mathbb{RP}^3 \times \mathbb{RP}^3; \mathbb{Z})$.

⇒ get:

i	$H_i(\mathbb{RP}^3 \times \mathbb{RP}^3; \mathbb{Z})$
0	$\mathbb{Z}$
1	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
2	$\mathbb{Z}/2$
3	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$
4	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
5	0
6	$\mathbb{Z}$