Künneth for cohomology:

implicitly R-coeffs.

Observe that the map $\Theta : C_\bullet(X\times Y) \longrightarrow C_\bullet(X)\otimes C_\bullet(Y)$ (from E-Z theorem) induces, by dualizing, a map

$$\mathrm{Hom}_R(C_\bullet(X)\otimes C_\bullet(Y), R) \xrightarrow{\Theta^*} \mathrm{Hom}_R(C_\bullet(X\times Y), R) = C^\bullet(X\times Y; R).$$
$$\Phi \longmapsto \Phi\circ\Theta.$$

this is not necessarily equal to $C^\bullet(X;R)\otimes C^\bullet(Y;R) = \mathrm{Hom}_R(C_\bullet(X),R)\otimes \mathrm{Hom}_R(C_\bullet(Y),R)$.

Using the fact that R is a ring, can define for any two R-modules M, N a map

$$\mathrm{Hom}_R(M,R)\otimes \mathrm{Hom}_R(N,R) \longrightarrow \mathrm{Hom}(M\otimes N, R),$$
$$(f, g) \longmapsto \underbrace{\{ m\otimes n \longmapsto \underset{R\otimes R \xrightarrow[\text{mult.}]{\cong} R}{f(m)\otimes g(n)} \longmapsto f(m)\cdot g(n) \}}_{m: R\times R\to R}.$$

we'll call this $m\circ(f\otimes g)$, or just $\underline{f\otimes g}$:

Using this, we get a map

$$\begin{array}{ccccc} \mathrm{Hom}_R(C_\bullet(X),R)\otimes \mathrm{Hom}_R(C_\bullet(Y),R) & \longrightarrow & \mathrm{Hom}_R(C_\bullet(X)\otimes C_\bullet(Y), R) & \xrightarrow{\Theta^*} & \mathrm{Hom}_R(C_\bullet(X\times Y),R) \\ (f,g) & \longmapsto & f\otimes g & \longmapsto & (f\otimes g)\circ\Theta \\ \| & & & & \| \\ C^\bullet(X;R)\otimes C^\bullet(Y;R) & & \longrightarrow & & C^\bullet(X\times Y;R) \end{array}$$

Def: call this map the $\underline{\text{cohomology cross product}}$, $\times$.

$\underline{\text{Lemma}}$ (omitted): $\delta(f\times g) = \underline{\delta f\times g + (-1)^{\deg(f)} f\times \delta g}$.

(Rmk: for the above to be true w/ signs, use a different convention $\delta f = (-1)^{\deg(f)+1} f\circ\partial$).
(rather than $\delta f = f\circ\partial$)

Also, $\times$ is natural with respect to maps $f: X\to X'$, $g: Y\to Y'$ (exercise: spell out)
& canonical (ind. of choice of Θ) — follows from analogous statement for Θ:
up to chain homotopy — up to chain homotopy!

$\underline{\text{Rmk}}$: there's a canonical element $1\in C^0(X;R)$ defined by $1(x:\Delta^0\to X) := 1\in R$,
(i.e., constant function).

This can be thought of as pulled back from $1\in C^0(\{pt\};R)$ $(1(\{pt.\}) = 1)$, via $X\xrightarrow{\varepsilon} pt$, i.e., $\varepsilon^* 1_{pt} = 1_X$.

Claim: Given X, Y spaces, recall have $\pi_X : X \times Y \to X$, $\pi_Y : X\times Y \to Y$, and
$-\times- : C^\bullet(X) \otimes C^\bullet(Y) \to C^\bullet(X \times Y)$, then

$\alpha \times 1_Y = \pi_X^* \alpha$, and similarly $1_X \times \beta = \pi_Y^* \beta$. $\{\varepsilon : Y \to pt.\}$

To see e.g., ↻, we'll make use of the fact that $1_Y = \varepsilon^* 1_{pt}$; so by naturality

$$\alpha \times 1_Y = \alpha \times \varepsilon^* 1_{pt} \underset{\text{naturality}}{=\!=} (id_X, \varepsilon)^* (\alpha \times 1_{pt})$$

$$\begin{tikzcd} X \times Y \arrow[r, "(id_X,\varepsilon)"] \arrow[dr, "\pi_X"'] & X \times pt. \arrow[d, "\cong"] \\ & X \end{tikzcd}$$

(the identification $X \times pt \cong X$ sends $\alpha \times 1_{pt} \leftrightarrow \alpha$).

$$= \pi_X^*(\alpha).$$

X, Y spaces.

Thm (Künneth for cohomology): If $R = \Lambda$ is a field and $H_i(X)$ is finite-rank for each i ("X is finite type over Λ") (or Y can be instead of X), then the cross product induces an isomorphism

$$-\times- : H^\bullet(X;\Lambda) \otimes H^\bullet(Y;\Lambda) \xrightarrow{\cong} H^\bullet(X \times Y;\Lambda).$$

(this also holds over a ring R as stated provided each $H_i(X)$ is a finitely generated, free R-module)

Sketch of proof: Over a field, UCT simplifies & gives:

$$H^\bullet\big(Hom_\Lambda(C_\bullet(X) \otimes C_\bullet(Y), \Lambda)\big) \xrightarrow[\text{UCT (coh.)}]{\cong} Hom_\Lambda\big(H_\bullet(\underbrace{C_\bullet(X) \otimes C_\bullet(Y)}_{\cong_{EZ}\, C_\bullet(X \times Y)}), \Lambda\big)$$

$$\downarrow \text{alg. Künneth (also simplifies)}$$

$$Hom(H_\bullet(X) \otimes H_\bullet(Y), \Lambda). \quad \cong Hom(H_\bullet(X \times Y), \Lambda) \underset{\cong}{\uparrow} H^\bullet(X \times Y, \Lambda).$$

Also, simplified alg. Künneth implies

$$H^\bullet\big(C^\bullet(X;\Lambda) \otimes C^\bullet(Y;\Lambda)\big) \xleftarrow{\cong} H^\bullet(X;\Lambda) \otimes H^\bullet(Y;\Lambda).$$

All together by using these isomorphisms, we can show the cohomological cross product factors as:

$$\begin{tikzcd} H^\bullet(X;\Lambda) \otimes H^\bullet(Y;\Lambda) \arrow[rr, "\times"] \arrow[d, "\cong"] & & H^\bullet(X \times Y;\Lambda) \arrow[d, "\cong"] \\ Hom_\Lambda(H_\bullet(X), \Lambda) \otimes Hom_\Lambda(H_\bullet(Y), \Lambda) & & Hom(H_\bullet(X \times Y), \Lambda) \end{tikzcd}$$

uses simplified UCT

$$\xrightarrow[\text{(mult.)}]{\otimes} \operatorname{Hom}(H_\bullet(X)\otimes H_\bullet(Y), \Lambda) \xrightarrow[\cong]{[\theta]^*}$$

uses simplified homology-Künneth.

Therefore $\times$ is an isomorphism iff $\otimes$ is.

Obs: $\otimes : \operatorname{Hom}_\Lambda(V, \Lambda) \otimes \operatorname{Hom}_\Lambda(W, \Lambda) \xrightarrow{(*)} \operatorname{Hom}(V\otimes W, \Lambda)$ is an iso. if one of V, W are finite rank in each degree.

graded vector spaces

(exercise).

notation: $Z^* := \operatorname{Hom}(Z, \Lambda)$

Note: RHS $\cong \operatorname{Hom}(V, W^*)$, & LHS $= V^* \otimes W^*$, & $(*)$ is the canonical map $V^* \otimes Q \to \operatorname{Hom}(V, Q)$ for $Q = W^*$.

Exercise: How can it fail if both V, W infinite dimensional? (e.g., case $V = W = \Lambda^\infty = \bigoplus_{i=1}^{\infty} \Lambda$.)

The cup product on cohomology

Recall any top. space X has a diagonal map $\Delta : X \longrightarrow X \times X$, $x \longmapsto (x, x)$.

On homology, we could use this to get a "coproduct":

$$\begin{tikzcd} C_\bullet(X) \arrow[r, "\Delta_\#"] \arrow[rr, bend right, "\text{also called } \Delta_\#"'] & C_\bullet(X \times X) \arrow[r, "\cong", "\theta"'] & C_\bullet(X) \otimes C_\bullet(X). \end{tikzcd}$$

Def: A diagonal approximation is a chain map $C_*(X) \to C_*(X) \otimes C_*(X)$ $\forall X$, natural in X, which in degree 0 sends $x_0 \longmapsto x_0 \otimes x_0$.

Thm: Any two diagonal approximations are ch. htopic.

(Pf same as previous 'method of acyclic models' proofs).

Could think of such a $\Delta_\#$ as inducing a coproduct on homology, or dually:

Def: The cup product on singular co-chains (w/ arbitrary coeffs. in some R), denoted $\cup$, is defined as;

$$\begin{tikzcd} C^\bullet(X;R) \otimes C^\bullet(X;R) \arrow[r, "\times"] \arrow[dr, dashed, "\text{mult. } (\otimes)"'] & C^\bullet(X\times X;R) \arrow[r, "\Delta^\#\ (=(\Delta_\#)^*)"] & C^\bullet(X;R). \\ & \operatorname{Hom}(C_\bullet(X)\otimes C_\bullet(X); R) \arrow[u, dashed, "\theta^*"] \arrow[ur, dashed, "(\Delta_\#)^*"'] & \end{tikzcd}$$

↖ diagonal approximation

"$\alpha \cup \beta$"

cup product

Since $- \times -$ and $\Delta^{\#}$ are (co)-chain maps, $\cup$ is too, hence it induces a cohomology-level map, also call $\cup$ (by abuse of notation)

Thm: (properties of the cup product on <u>cohomology</u>)

(1) $\cup$ is natural, meaning if $f: X \to Y$, then $f^*(\alpha \cup \beta) = (f^*\alpha) \cup (f^*\beta)$.

(exercise: immediate (b/c $\times$ is natural, and $f: X \to Y$ induces $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Delta_X \downarrow & & \downarrow \Delta_Y \\ X \times X & \xrightarrow{(f,f)} & Y \times Y \end{array}$))

(2) $\alpha \cup 1 = \alpha = 1 \cup \alpha$ for any α. (the element 1 is a <u>unit</u> for $\cup$).

(3) $\alpha \cup (\beta \cup \gamma) = (\alpha \cup \beta) \cup \gamma$. (associativity)

these will follow on chain-level from a particular chain model of $\cup$ (though both can be proved without this)

(4) $\alpha \cup \beta = (-1)^{\deg(\beta)\deg(\alpha)} \beta \cup \alpha$. ← on cohomology! (graded commutativity)

(typically cannot realize commutativity on chain level over arbitrary R).

(5) If (X,A) pair, and $i: A \hookrightarrow X$, so $i^*: H^*(X) \to H^*(A)$, $\delta: H^p(A) \to H^{p+1}(X,A)$ connecting map, then

$$\delta(\underbrace{\alpha \cup i^*(\beta)}_{H^*(A)}) = \underbrace{\delta(\alpha) \cup \beta}_{H^*(X,A)}.$$

← uses the fact that $\cup$ defines a product on relative co-chains $C^*(X,A) \times C^*(X,A) \to C^*(X,A)$.

To prove commutativity, we'll make use of the following lemma:

<u>Lemma:</u> Let $T: X \times Y \to Y \times X$ be the factor reversing map $T(x,y) = (y,x)$.

& for chain complexes $C_\bullet, D_\bullet$, let $\tau: C_\bullet \otimes D_\bullet \to D_\bullet \otimes C_\bullet$, $(c,d) \longmapsto (-1)^{\deg(c)\deg(d)} d \otimes c$ factor-reversing chain map.

Then the following diagram is <u>homotopy-commutative</u>:

$$\begin{tikzcd} C_\bullet(X \times Y) \arrow[r, "\Theta_{(X,Y)}"] \arrow[d, "T_\#"] & C_\bullet(X) \otimes C_\bullet(Y) \arrow[d, shift left, "\tau"] \\ C_\bullet(Y \times X) \arrow[r, "\Theta_{(Y,X)}"] & C_\bullet(Y) \otimes C_\bullet(X) \arrow[u, shift left, "\tau"] \end{tikzcd}$$

<u>Pf:</u> Consider $\tau \circ \Theta \circ T_\#$ and $\Theta: C_\bullet(X \times Y) \to C_\bullet(X) \otimes C_\bullet(Y)$. These are both natural maps, chain maps, & agree in degree 0 $\Longrightarrow$ $\tau \circ \Theta \circ T_\#$ and Θ are chain homotopic via some chain homotopy G. (by Uniqueness Thm, using

acyclic models).

$$\Rightarrow \tau \circ \tau \circ \theta \circ T_\# \text{ and } \tau \circ \theta \text{ are chain htpic, via } H = \tau \circ G.$$

$$\overset{\shortparallel}{\theta \circ T_\#}$$