Last time:

Lemma: let $T: X \times Y \to Y \times X$ be the factor reversing map $T(x,y) = (y,x)$.

& for chain complexes $C_\bullet, D_\bullet$, let $\tau: C_\bullet \otimes D_\bullet \to D_\bullet \otimes C_\bullet$, $(c,d) \longmapsto (-1)^{\deg(c)\deg(d)} d \otimes c$ factor-reversing chain map.

Then the following diagram is homotopy-commutative:

$$\begin{array}{ccc} C_\bullet(X \times Y) & \xrightarrow{\theta_{(X,Y)}} & C_\bullet(X) \otimes C_\bullet(Y) \\ \downarrow T_\# & & \uparrow \tau \quad \downarrow \tau \\ C_\bullet(Y \times X) & \xrightarrow{\theta_{(Y,X)}} & C_\bullet(Y) \otimes C_\bullet(X). \end{array}$$

proven with this, therefore by applying τ again, get htpy commutative diagram with (*)

Now, this implies:

$$C_\bullet(X) \xrightarrow{\Delta_\#} C_\bullet(X \times X) \xrightarrow{\theta} C_\bullet(X) \otimes C_\bullet(X) \xrightarrow[\text{factor reverse}]{\tau} C_\bullet(X) \otimes C_\bullet(Y)$$

is chain homotopic to

$$C_\bullet(X) \xrightarrow[\Delta_\#]{} C_\bullet(X \times X) \xrightarrow[T_\#]{} C_\bullet(X \times X) \xrightarrow{\theta} C_\bullet(X) \otimes C_\bullet(X), \text{ i.e., to}$$

$(T \circ \Delta)_\# = \Delta_\#$ ← why? $X \xrightarrow{\Delta} X \times X \xrightarrow{T} X \times X$, $x \longmapsto (x,x) \longmapsto (x,x)$,

$$C_\bullet(X) \xrightarrow{\theta \circ \Delta_\#} C_\bullet(X) \otimes C_\bullet(X) \qquad \text{(the usual coproduct)}$$

Dualizing, we see that for $\alpha, \beta \in C^\bullet(X)$ (R-coeffs.), $a \in C_\bullet(X)$.

$$\alpha \cup \beta\,(a) := \overbrace{\alpha \otimes \beta}^{\times}\,(\underbrace{\theta(\Delta_\# a)}_{\in\, C_\bullet(X) \otimes C_\bullet(X)}) \simeq \alpha \otimes \beta\,(\tau \circ \theta \circ \Delta_\#(a))$$

implicitly applying α to first factor, β to second, & multiplying result.

chain homotopic via dualizing above

$$(-1)^{\deg(\beta)\deg(\alpha)} \beta \otimes \alpha\,(\theta \circ \Delta_\#(a)) = (-1)^{\deg(\alpha)\deg(\beta)} \beta \cup \alpha\,(a). \qquad \blacksquare.$$

⟹ cup product is commutative ..

It will help to have an explicit formula for Θ:

Alexander-Whitney map Θ_{AW}

Let $\Delta^n = [e_0, \dots, e_n]$ be the standard simplex. For any $0 \le p \le n$, $0 \le q \le n$,

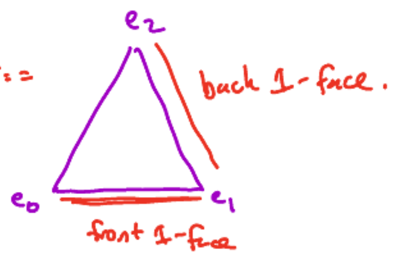
Define the front p-face of Δ^n to be $[e_0, \dots, e_p]$ inside $[e_0, \dots, e_n]$, or via:

$$
\begin{aligned}
f_p : \Delta^p &\hookrightarrow \Delta^n \\
e_0 &\longmapsto e_0 \\
&\vdots \\
e_p &\longmapsto e_p .
\end{aligned}
$$

Define the back q-face of Δ^n to be $\underset{\Delta^q}{[e_{n-q}, \dots, e_n]} \hookrightarrow \Delta^n$.

$$
\begin{aligned}
g_q : \Delta^q &\hookrightarrow \Delta^n \\
e_0 &\longmapsto e_{n-q} \\
&\vdots \\
e_q &\longmapsto e_n .
\end{aligned}
$$

e.g., $\Delta^2 =$



Using this, let's define an explicit version of Θ,

$$
\Theta_{AW} : C_n(X \times Y) \longrightarrow \big(C_\bullet(X) \otimes C_\bullet(Y)\big)_n = \bigoplus_{i=0}^{n} C_i(X) \otimes C_{n-i}(Y)
$$

"Alexander-Whitney"

by:

$$
\{ \underset{(\pi_X \sigma, \pi_Y \sigma)}{\sigma} : \Delta^n \to X \times Y \} \longmapsto \sum_{i=0}^{n} \underbrace{(\pi_X \sigma \circ f_i)}_{\substack{\pi_X \sigma|_{[e_0, \dots, e_i]} \\ \in C_i(X)}} \otimes \underbrace{(\pi_Y \sigma \circ g_{n-i})}_{\substack{\pi_Y \sigma|_{[e_i, \dots, e_n]} \\ \in C_{n-i}(Y)}}
$$

(q: check signs of above)

Lemma: Θ_{AW} is natural in X, Y, is a chain map, and coincides w/ any other Θ in degree 0.

$\Rightarrow \Theta_{AW}$ is a choice of Θ (hence ch. htpic to any other choice).

Pf idea: Naturality is straightforward, explicitly need to compute
$$\partial\, \Theta_{AW}(\sigma) \stackrel{?}{=} \Theta_{AW}(\partial \sigma). \qquad \text{(exercise)}.$$

Using this, define $\Theta_{AW} \circ \Delta_{\#}$ (homological coproduct/diagonal approximation) — chain model of

$$\{\tau : \Delta^n \to X\} \stackrel{\Delta_\#}{\longmapsto} \{\Delta_\# \tau = (\tau,\tau) : \Delta^n \to X \times X\} \stackrel{\Theta_{AW}}{\longmapsto}$$
$$\sum_{i=0}^{n} \tau|_{[e_0,\dots,e_i]} \otimes \tau|_{[e_i,\dots,e_n]}.$$

and $\alpha \cup \beta$ can be given the model:

$$\boxed{\underset{\deg p}{\alpha} \cup \underset{\deg q}{\beta} (\underset{\deg r = p+q}{\tau})} = \alpha \otimes \beta (\Theta_{AW} \circ \Delta_\# (\tau))$$
$$= \alpha \otimes \beta \Big(\sum_{i=0}^{r} \tau|_{[e_0,\dots,e_i]} \otimes \tau|_{[e_i,\dots,e_n]} \Big)$$

α (i-simplex) $= 0$ unless $i = \deg(\alpha)$

$$\boxed{:= (-1)^{pq} \alpha(\tau|_{[e_0,\dots,e_p]}) \cdot \beta(\tau|_{[e_p,\dots,e_{n=p+q}]}).}$$

(b/c $f \otimes g (a \otimes b) := (-1)^{\deg(g)\deg(a)} f(a)\, g(b)$).

in sense that $1 \cup \alpha = \alpha \cup 1 = \alpha$

Exercise: This cochain model for cup product is associative + unital on chain level (by direct computation).

$\Rightarrow$ (since any two models of Θ_{AW} induce same $\cup$ on cohomology) $\cup$ is associative & unital on cohomology. (even if it may not be on chain level for a different Θ).

Compatibility with cross product

X, Y spaces, R coefficient ring (implicit), have

- $\times : H^\bullet(X) \otimes H^\bullet(Y) \longrightarrow H^\bullet(X \times Y)$

• $\Delta_X : X \to X\times X$, $\Delta_Y : Y \to Y\times Y$, $\Delta_{X\times Y} : X\times Y \to X\times Y\times X\times Y$.

Obs: $\Delta_{X\times Y} = T\circ(\Delta_X, \Delta_Y)$ where $T : X\times X\times Y\times Y \longrightarrow X\times Y\times X\times Y$

swaps 2nd + 3rd factors.

Lemma: $(\underset{H^*(X)}{\alpha_1}\times\underset{H^*(Y)}{\beta_1}) \cup (\alpha_2\times\beta_2) = (-1)^{\deg(\alpha_2)\deg(\beta_1)}(\alpha_1\cup\alpha_2)\times(\beta_1\cup\beta_2)$

Pf: $LHS = \Delta^*_{X\times Y}\left((\alpha_1\times\beta_1)\times(\alpha_2\times\beta_2)\right)$

$\overset{obs}{=} (\Delta_X, \Delta_Y)^* T^*(\alpha_1\times\beta_1 \times \alpha_2\times\beta_2)$

erase parentheses b/c $\times$ is assoc. (on cohomology) (on chain level, associators may cost chain homotopy ~~...~~ unless using θ_{AW}).

$= (-1)^{\deg(\alpha_2)\deg(\beta_1)}(\Delta_X, \Delta_Y)^*(\alpha_1\times\alpha_2\times\beta_1\times\beta_2)$

$= RHS.$ ∎

Cor: The Künneth isomorphism (which holds when one of X, Y is of 'finite type'):

$$H^*(X)\otimes H^*(Y) \xrightarrow{\times} H^*(X\times Y)$$

is a ring iso.

(LHS has a ring str. by $(x\otimes y)\cdot(x'\otimes y') := (-1)^{\deg(y)\deg(x')}(x\cup x')\otimes(y\cup y')$.)

Cor: ($\times$ in terms of $\cup$). For any $\alpha\in H^*(X)$, $\beta\in H^*(Y)$

$\pi_X : X\times Y \to X$
$\pi_Y : X\times Y \to Y$.

$$\boxed{\alpha\times\beta} \underset{\substack{\text{Lemma}\\ \text{(\& unitality)}}}{=} (\alpha\times 1_Y)\cup(1_X\times\beta) \underset{\substack{\text{last}\\ \text{time}}}{=} \boxed{(\pi_X^*\alpha)\cup(\pi_Y^*\beta)}$$

Rmk: RHS is how Hatcher defines $\times$, at least initially.

Example: (1) compute $H^*(S^2\times S^4)$.

We have $H^*(S^{2k}) = \begin{cases} \mathbb{Z} & \deg 2k \\ \mathbb{Z} & \deg 0 \\ 0 & \text{else} \end{cases}$ (by UCT).

Let's denote the degree $2k$ generator by α_{2k}. Note $\alpha_{2k}\cup\alpha_{2k} = 0$ b/c $H^{4k}(S^{2k}) = 0$.

So, as a ring, $H^\bullet(S^{2k}) \cong \mathbb{Z}[\alpha_{2k}]/\alpha_{2k}^2$, $\text{degree}(\alpha_{2k}) = 2k$.
$|\alpha_{2k}|$

meaning $\bigoplus_i H^i(S^{2k})$

Therefore by Künneth (works over $\mathbb{Z}$ b/c $H^\bullet(S^{2k})$ free finite type),

$$H^\bullet(S^2 \times S^4) \cong \mathbb{Z}[\alpha_2]/\alpha_2^2 \otimes \mathbb{Z}[\alpha_4]/\alpha_4^2 \cong \mathbb{Z}[\alpha,\beta]/\alpha^2,\beta^2,$$

$|\alpha| = 2$, $|\beta| = 4$.

Note $\alpha \bullet \beta$ generates in degree 6.
($\alpha = pr_{S^2}^* \alpha_2$, $\beta = pr_{S^4}^* \alpha_4$).

(2) $T^n = (S^1)^n$. By same reasoning as above $H^\bullet(S^1;\mathbb{Z}) \cong \mathbb{Z}[\theta]/\theta^2$ $|\theta| = 1$.
$\cong \mathbb{Z}[\theta]$

(modding by θ^2 is redundant if we're imposing graded commutativity over $\mathbb{Z}$:
$$\theta \smile \theta = (-1)^{\deg(\theta)\deg(\theta)} \theta \smile \theta = -\theta \smile \theta \underset{\text{over } \mathbb{Z}}{\Longrightarrow} \theta \smile \theta = 0)$$

Therefore $H^\bullet(T^n) \cong \mathbb{Z}[\theta_1, \dots, \theta_n]$ "exterior algebra in n-variables"
(w/ $\theta_i^2 = 0$ implicit). (sometimes $\mathbb{Z}|\theta_1, \dots, \theta_n|$ or —)

each $|\theta_i| = 1$ $\theta_i = (\pi_i^*)(\theta)$.

$j \neq i$: $\theta_i \cdot \theta_j = -\theta_j \cdot \theta_i$. (by graded commutativity).

note $H^n(T^n) = \mathbb{Z}$ gen. by $\theta_1 \cdot \theta_2 \cdots \theta_n$, & $rk_{\mathbb{Z}} H^i(T^n) = \binom{n}{i}$.

Ex: $S^2 \vee S^4$. know: (by UCT)
$$H^k(S^2 \vee S^4) = \begin{cases} \mathbb{Z} & \deg 4, \text{ gen. by } j_{S^4}^* \alpha_4 = x_2 \\ \mathbb{Z} & \deg 2, \text{ gen. by } j_{S^2}^* \alpha_2 = x_1 \\ \mathbb{Z} & \deg 0 \end{cases}$$

$j_{S^2}: S^2 \vee S^4 \longrightarrow S^2$ projection (collapse S^4 to point) $\quad j_{S^4}: S^2 \vee S^4 \to S^4$

check: $j_{S^2}^*: H^2(S^2) \longrightarrow H^2(S^2 \vee S^4)$ is an isomorphism (exercise)

Q: is there a relation in H^4 between x_1^2 and x_2?

by naturality, $x_1 \cup x_1 = \int_{S^2} \alpha_2 \cup \int_{S^2} \alpha_2 \underset{\text{naturality}}{=} \int_{S^2} (\underbrace{\alpha_2 \cup \alpha_2}_{H^4(S^2)=0.}) = \int_{S^2}(0) = 0.$

So, $\boxed{\text{no}}$.

by above. b/c $H^8 = 0$ b/c $H^6 = 0$.

Hence, $H^*(S^2 \vee S^4) \cong \mathbb{Z}[x_1, x_2] / x_1^2, x_2^2, x_1 x_2$.

deg 2 deg 4.