The cup product on relative co-chains:

X space, $A, B \underset{\text{open}}{\subset} X$.

We know $C^n(X,A) = \mathrm{Ann}(C_n(A)) \subset \mathrm{Hom}(C_n(X), R)$ (w/ R-coeffs.)

If $\phi \in C^p(X,A)$, $\psi \in C^q(X,B)$, where $p+q=n$,

$$\phi \smile \psi(\underset{\uparrow\, C_n}{\sigma}) = \pm\, \phi(\sigma|_{[e_0,\dots,e_p]}) \cdot \psi(\sigma|_{[e_p,\dots,e_{n=p+q}]})$$

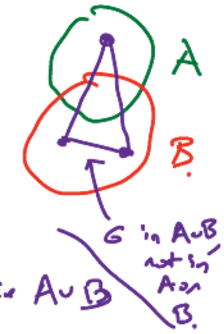
is zero if $\mathrm{im}(\sigma) \subset A$ or if $\mathrm{im}(\sigma) \subset B$ entirely, so $\phi \smile \psi \in \mathrm{Ann}(C_n(A) + C_n(B))$,
(b/c then its front p-face is too) (b/c then its back face is too).

we'll use the shorthand $\mathrm{Ann}(C_n(\text{"}A+B\text{"})) \overset{\text{def}}{=} \mathrm{Ann}(C_n(A)+C_n(B))$.

$$\begin{array}{c} \| \\ C^n(X, \text{"}A+B\text{"}) \overset{\text{not nec.}}{\neq} C^n(X, A\cup B) \end{array}$$

∃ natural map (incl.)

annihilates simplices in A or in B. annihilates simplices in $A \cup B$



We'll also abbreviate $C_n(A+B) := C_n(A) + C_n(B)$ (sum in $C_n(X)$). We have a natural inclusion.

$i: C_n(A+B) \hookrightarrow C_n(A\cup B)$, which we note (by prev. semester) induces an iso. on homology $H_n(C_\bullet(A+B)) \xrightarrow{\cong} H_n(C_\bullet(A\cup B))$.

(more generally, barycentric subdiv. $\Rightarrow$ for any Y w/ a cover $\mathcal{U} = \{U_i\}$, ($Y = A\cup B$ w/ cover $\{A,B\}$)

$$C_\bullet^{\mathcal{U}}(Y) \xrightarrow[\text{on } H_\bullet]{\cong} C_\bullet(Y)\,).$$

chains supported in some U_i

$\Rightarrow$ induces an iso. on cohomology $H^n(A\cup B) \xrightarrow{\cong} H^n(\text{"}A+B\text{"})$.

By comparing LES of pair $(X, A\cup B)$ w/ pair $(X, \text{"}A+B\text{"})$ in cohomology, we can deduce (exercise) that the canonical map $C^\bullet(X, A\cup B) \xrightarrow{k} C^\bullet(X, \text{"}A+B\text{"})$ induces a cohomology iso,

$$[k]: H^n(X, A\cup B) \xrightarrow{\cong} H^n(X, \text{"}A+B\text{"}).$$

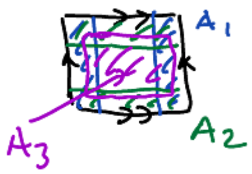
$$(\;—\;) \longmapsto [\phi \smile \psi]$$

Cor: Get a cup product map $[\smile] : (= [k]^{-1} \circ (\smile)) : H^p(X,A) \otimes H^q(X,B) \to H^{n=p+q}(X, A\cup B)$.

Exercise on HW (using this): show that if X is covered by m acyclic open sets then all m-fold cup products of pos. degree classes are zero.

e.g.,

$\Rightarrow$ any α, β pos. degree coh.
$\Rightarrow \alpha \cup \beta = 0$.



$\Rightarrow$ In T^2 given $\alpha_1, \alpha_2, \alpha_3$ pos. degree, $\alpha_1 \cup \alpha_2 \cup \alpha_3 = 0$ in cohomology.

Conversely, we'll compute $H^\bullet(T^2)$ has a nontrivial cup product of degree 1-classes
$\Rightarrow T^2 \neq A \cup B$, A, B contractible.

Important examples of cohomology rings:

$\underbrace{\mathbb{R}P^n}_{\text{over } R=\mathbb{Z}/2}$, $\underbrace{\mathbb{C}P^n, \mathbb{H}P^n}_{\text{over } R=\mathbb{Z}}$, $n \in \mathbb{N} \cup \{\infty\}$. (via e.g., $\mathbb{C}P^\infty = \bigcup_{n}^{n=\infty} \mathbb{C}P^n$, where $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2 \hookrightarrow \cdots$)

We know from studying cellular chain complexes (+UCT) that

$$H^k(\mathbb{R}P^n; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & k=0,\dots,n \\ 0 & \text{else.} \end{cases}, \quad \text{Similarly } H^k(\mathbb{C}P^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & k=0,2,\dots,2n \\ 0 & \text{else.} \end{cases}$$

$$H^k(\mathbb{H}P^n; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k=0,4,\dots,4n \\ 0 & \text{else.} \end{cases}$$

Write. $h \in H^1(\mathbb{R}P^n; \mathbb{Z}/2)$ generator of H^1.
(since over $\mathbb{Z}/2$, $h \cup h$ need not be zero).

Thm: $h^k := \underbrace{h \cup \cdots \cup h}_{k \text{ times}}$ is a generator for $H^k(\mathbb{R}P^n; \mathbb{Z}/2)$, for any $k \le n$ (if $n < \infty$)

i.e., $\boxed{H^\bullet(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2\,[h] / h^{n+1}.}$ truncated polynomial alg.

$|h| = 1$.

Thm: If $h \in H^2(\mathbb{C}P^n; \mathbb{Z})$ is a generator for H^2, then h^k generates $H^{2k}(\mathbb{C}P^n; \mathbb{Z})$ for all $k = 1, \dots, n$.

$\Rightarrow H^\bullet(\mathbb{C}P^n) \cong \mathbb{Z}[h] / h^{n+1}$, $|h| = 2$.

$\Rightarrow H^*(\mathbb{CP}^n) \cong \mathbb{Z}[h]/h^{n+1}$. $|h| = 2$

similarly for $\mathbb{HP}^n$, $|h| = 4$.

We'll prove these theorems later, as a consequence of other results (e.g. Poincaré duality)

<u>But we can already explore some consequences</u>:

- Observe that $H^i(\mathbb{CP}^3;\mathbb{Z}) \cong H^i(S^2 \times S^4;\mathbb{Z})$ in every degree (similarly on H_i).

  ($\mathbb{Z}$ in degree $0, 2, 4, 6$, 0 otherwise)

  However, the ring structures on H^* are different:

  H^* $\quad \mathbb{Z}[h]/h^4 \quad$ vs. $\quad \mathbb{Z}[\alpha,\beta]/\alpha^2,\beta^2 \quad |\alpha| = 2, |\beta| = 4.$

  $|h| = 2$

  $\mathbb{CP}^3 \qquad\qquad S^2 \times S^4$

  are not isomorphic rings, so $\mathbb{CP}^3 \not\simeq S^2 \times S^4$ up to homotopy equivalence

  (e.g., any homotopy equivalence would send h to $\pm\alpha$, but $h^2 \neq 0$, and $\alpha^2 = 0$ $\Rightarrow\!\Leftarrow$).

- Look at $\mathbb{CP}^2$ vs. $S^2 \vee S^4$.

  <u>Note</u>: $\mathbb{CP}^2 = \underbrace{e^0 \cup e^2}_{\mathbb{CP}^1 = S^2} \cup e^4 \qquad S^2 \vee S^4 = \underbrace{e^0 \cup e^2}_{S^2} \cup e^4$

  If $\mathbb{CP}^2 \underset{\text{h.e.}}{\not\simeq} S^2 \vee S^4$, we conclude the attaching maps $f_{\mathbb{CP}^2} : \partial e^4 = S^3 \longrightarrow \overbrace{S^2}^{\text{3-skeleton}}$ and

  $f_{S^2 \vee S^4} : \partial e^4 = S^3 \xrightarrow[\text{const. at } e^0]{} S^2$

  cannot be homotopic.

  Let's check H^* rings: $H^*(\mathbb{CP}^2) \cong \mathbb{Z}[h]/h^3 \quad |h| = 2.$

  $H^*(S^2 \vee S^4) \cong \mathbb{Z}[\alpha,\beta]/\alpha^2,\beta^2,\alpha\beta \quad |\alpha| = 2, |\beta| = 4$

  so the attaching map: $S^3 \to S^2$ ('Hopf map') used to construct $\mathbb{CP}^2$ represents a non-trivial homotopy class. $(*)$.

Define <u>higher homotopy groups</u>

$$\pi_k(X,x_0) := [(S^k,*),(X,x_0)]\}.$$

($\pi_0 := \{$connected components$\}$, π_1 = usual fundamental group).

(*) implies:

<u>Cor</u>: $\pi_3(S^2) \neq 0$.

<u>The cap product</u>

We've introduced $H^\bullet(X)$ & showed it has structure of (graded) comm. ring, via cup product.

It turns out that $H_\bullet(X)$ has the structure of a (graded) module over $H^\bullet(X)$, via an operation called the <u>cap product</u>.

Fundamentally, on the chain level, the cap product is induced by the same information as the cup product, namely the homological coproduct/diagonal approximation:

$$\Delta: C_\bullet(X) \longrightarrow C_\bullet(X) \otimes C_\bullet(X)$$

defined by $\Delta := \Theta \circ \Delta_\#$

the map $C_\bullet(X) \to C_\bullet(X \times X)$ induced by $\Delta: X \to X \times X$

the inverse to EZ: $C_\bullet(X \times X) \xrightarrow{\simeq} C_\bullet(X) \otimes C_\bullet(X)$.

<u>Def'n</u>: Given $\alpha \in C^p(X)$, $\beta \in C_q(X)$,

define $\alpha \frown \beta := id \otimes \alpha(\underbrace{\Delta\beta})$

lives in $(C_\bullet(X) \otimes C_\bullet(X))_q := \bigoplus_{i+j=q} C_i(X) \otimes C_j(X)$

$\Delta\beta := \sum_{i+j=q} (\Delta\beta)_{i,j}$

recall $\alpha: C_p(X) \to R$,

extend to $\alpha: C_\bullet(X) \to R$ by saying $\alpha(C_i(X)) = 0$ for $i \neq p$.

i.e., $\alpha \frown \beta = id \otimes \alpha(\Delta\beta_{q-p,p}) \in C_{q-p}(X)$,

b/c $id \otimes \alpha$ is only non-zero on this piece.

If we use the Alexander-Whitney model of homological coproduct

$$\Delta_{AW} := \Theta_{AW} \circ \Delta_\# : \underset{\text{degree } q}{\sigma} \longmapsto \sum_{i+j=q} \sigma|_{[e_0,\dots,e_i]} \otimes \sigma|_{[e_i,\dots,e_{i+j=q}]},$$

front i-face back j-face

we see that up to chain homotopy the cap product can be given the following chain-level model: linearly extend the following formula over all chains:

∃ a sign to apply α to this term

$$\underset{C^p(X;R)}{\alpha^p} \cap \underset{\substack{C_q(X;R) \\ \text{generator} \\ \sigma:\Delta^q\to X.}}{\sigma_q} := (id \otimes \alpha^p)\left(\underbrace{\sigma|_{[e_0,\dots,e_{q-p}]}}_{\text{front } q-p \text{ face}} \otimes \underbrace{\sigma|_{[e_{q-p},\dots,e_{q-p+p=q}]}}_{\text{back } p \text{ face.}} \right)$$

$$= (-1)^{p(q-p)}\, \sigma|_{[e_0,\dots,e_{q-p}]} \cdot \underbrace{\alpha(\sigma|_{[e_{q-p},\dots,e_p]})}_{\in R.}.$$

Prop: For any Θ (not just Θ_{AW}), the cap product satisfies the property of being a chain map

$$C^{-\bullet}(X) \otimes C_\bullet(X) \xrightarrow{\cap} C_\bullet(X) \quad \text{(of degree } 0\text{)}.$$

co-chains on X w/ degrees negated is a chain complex, in sense that δ decreases (−degree) by 1.

$C^p(X) \otimes C_q(X) \xrightarrow{\cap} C_{q-p}(X)$.
think of this as a degree $-p$ element of $C^{-\bullet}(X)$, then degrees are additive under cap product.

Namely, $\boxed{\partial(\alpha\cap\sigma) = \delta\alpha\cap\sigma + (-1)^p\alpha\cap\partial\sigma}$ (where $\alpha\in C^p(X)$)..

Hence, there is an induced map $H^\bullet(X)\otimes H_\bullet(X) \xrightarrow{\cap} H_\bullet(X)$, which (after negating degrees of cohomology groups) is a graded (degree 0) map.